Copyright
© School Curriculum and Standards Authority, 2015

This document apart from any third party copyright material contained in it may be freely copied, or communicated on an intranet, for non-commercial purposes in educational institutions, provided that the School Curriculum and Standards Authority is acknowledged as the copyright owner, and that the Authority's moral rights are not infringed.

Copying or communication for any other purpose can be done only within the terms of the Copyright Act 1968 or with prior written permission of the School Curriculum and Standards Authority. Copying or communication of any third party copyright material can be done only within the terms of the Copyright Act 1968 or with permission of the copyright owners.

Any content in this document that has been derived from the Australian Curriculum may be used under the terms of the Creative Commons Attribution-NonCommercial 3.0 Australia licence

Disclaimer
Any resources such as texts, websites and so on that may be referred to in this document are provided as examples of resources that teachers can use to support their learning programs. Their inclusion does not imply that they are mandatory or that they are the only resources relevant to the course.
Sample assessment tasks
Mathematics Methods – ATAR Year 12

Task 1 – Unit 3

Assessment type: Investigation

Materials required: Standard writing equipment
CAS calculator (to be provided by the student)

Other materials allowed: Nil

Total marks 30
Task weighting: 5%

This investigation develops students’ understanding of exact value and the concept of approximation and limiting values.

Notes for teachers:

- Students are expected to have completed Section 2.3 on differential calculus and should have used their calculator to demonstrate differentiation from first principles.

- Students are expected to have studied factorial arithmetic in Section 1.3 as well as the sum to infinity in Section 2.2.

- There are a number of methods to calculate the results required and it is assumed the students have studied at least one of these.

- Revision activities could be completed during class time or at home.
Task 1 – Unit 3 – Investigation – An interesting derivative

Question 1 (9 marks)

(a) Given \( y = a^x \) show \( \frac{dy}{dx} = \lim_{h \to 0} \left[ \frac{a^h - 1}{h} \right] a^x \) (3 marks)

(b) Using \( h = 0.0001 \), estimate a value of \( a \): \( 2 < a < 3 \), to 2 decimal places, such that \( \lim_{h \to 0} \left[ \frac{a^h - 1}{h} \right] \approx 1 \).

Complete the table below to show your working. (4 marks)

<table>
<thead>
<tr>
<th>( a )</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \left[ \frac{a^h - 1}{h} \right] )</td>
<td>0.6932</td>
<td>1.0987</td>
</tr>
</tbody>
</table>

(c) The solution to (b) suggests there is an exact value for \( a \) for which \( \left[ \frac{a^h - 1}{h} \right] = 1 \) exactly.

What is the implication of this for the derivative in part (a)? (1 mark)

(d) What is this exact value of \( a \)? (1 mark)

Given that \( 2! = 2 \times 1 \) and \( 3! = 3 \times 2 \times 1 \) and \( 4! = 4 \times 3 \times 2 \times 1 \)
then \( n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1 \)

Consider the following function, which is called a power series and which has an infinite number of terms:
\[
f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots + \frac{x^n}{n!} \ldots \text{ for } 0 \leq x \leq 1
\]
Question 2  

(a) Evaluate \( f(1) \) correct to 3 decimal places.  

(b) Evaluate \( f(0.1) \) correct to 3 decimal places.

You may have noticed the number of terms of the power series needed to achieve the required level of accuracy in parts (a) and (b) were not the same. Now compare the accuracy of the following truncated versions of \( f(x) \). Enter these functions into your calculator's function application and set up a table to view the values of the function needed to compare the level of accuracy.

Note: Also, the copy and paste function of the calculator will help make the task quicker.

\[

def_1(x) = 1 + x \\
def_2(x) = 1 + x + \frac{x^2}{2!} \\
def_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \\
def_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \\
def_5(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \\
def_6(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}
\]

(c) Calculate the following correct to 3 decimal places and state the least number of terms needed to do so:  

\( f(0.2) \)  
\( f(0.5) \)  
\( f(0.8) \)

(d) Comment on any pattern you saw from part (c).
Question 3 (11 marks)

Now consider the infinite series \( f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \)

(a) Evaluate the derivative of \( y = \frac{x^4}{4!} \) giving the answer in factorial form. (2 marks)

(b) Hence, evaluate the derivative of the infinite series \( f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \)
giving the answer in factorial form. (3 marks)

(c) Compare the result from Q3(b) above to the result in Q1(c). (1 mark)

(d) Compare \( f(1) \) from Q2(a) and the value for \( a \) where \( \frac{a^n-1}{h} = 1 \) from Q1(b). (1 mark)

(e) Using \( f(1) \) correct to 3 decimal places from Q2(a) and \( h=0.0001 \), evaluate the limit
\[
\lim_{h \to 0} \frac{a^h-1}{h}
\]
to 4 decimal places. (1 mark)

(f) Use the results from Q1–3 to make a conjecture regarding the two functions
\( y = a^x \) and \( f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \) (3 marks)

End of questions
Solutions and marking key for sample assessment task 1 – Unit 3

Question 1 (9 marks)

(a) Given \( y = a^x \) show \( \lim_{h \to 0} \frac{y}{x} = \left[ \frac{a^h - 1}{h} \right] a^x \) (3 marks)

Given \( y = a^x \) show

\[
\lim_{h \to 0} \frac{dy}{dx} = \lim_{h \to 0} \left\{ \frac{a^{x+h} - a^x}{h} \right\} = \lim_{h \to 0} \left\{ \frac{a^x(a^h - 1)}{h} \right\} = \lim_{h \to 0} \left\{ \frac{(a^h - 1)}{h} a^x \right\}
\]

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expresses the derivative from first principles</td>
<td>1</td>
<td>simple</td>
</tr>
<tr>
<td>Factorises with the denominator</td>
<td>1</td>
<td>simple</td>
</tr>
<tr>
<td>Rearranges the limit correctly</td>
<td>1</td>
<td>simple</td>
</tr>
</tbody>
</table>

(b) Using \( h = 0.0001 \) estimate a value of \( a \): \( 2 < a < 3 \), to 2 decimal places, such that \( \lim_{h \to 0} \left\{ \frac{a^h - 1}{h} \right\} \approx 1 \).

Complete the table below to show your working. (4 marks)

\[
a \approx 2.72 \Rightarrow \lim_{h \to 0} \left\{ \frac{a^h - 1}{h} \right\} \approx 1 \quad h = .0001
\]

<table>
<thead>
<tr>
<th>( a )</th>
<th>2</th>
<th>2.5</th>
<th>2.7</th>
<th>2.75</th>
<th>2.715</th>
<th>2.716</th>
<th>2.72</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{a^h - 1}{h} )</td>
<td>0.69317</td>
<td>0.9163</td>
<td>0.9933</td>
<td>1.0117</td>
<td>0.9988</td>
<td>0.9992</td>
<td>1.0007</td>
<td>1.09867</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Completes at least three values showing the limit approaching one</td>
<td>3</td>
<td>simple</td>
</tr>
<tr>
<td>Gives ( a ) correct to 2 decimal places</td>
<td>1</td>
<td>simple</td>
</tr>
</tbody>
</table>
(c) The solution to (b) suggests there is an exact value for $a$ for which \[ \frac{a^h - 1}{h} = 1 \] exactly.

What is the implication for this for the derivative in part (a)? (1 mark)

This implies that given $y = a^x$ and \[ \frac{dy}{dx} = \lim_{h \to 0} \left[ \frac{a^h - 1}{h} \right] a^x \] and \[ \lim_{h \to 0} \left[ \frac{a^h - 1}{h} \right] = 1 \]
for a given value of $a$ then $\frac{dy}{dx} = a^x$ or $\frac{d}{dx}(a^x) = a^x$.

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>States that the function is also its own derivative</td>
<td>1</td>
<td>complex</td>
</tr>
</tbody>
</table>

(d) What is this exact value of $a$? (1 mark)

\[ a = e \]

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>States that $a = e$ the natural base</td>
<td>1</td>
<td>complex</td>
</tr>
</tbody>
</table>

**Question 2** (10 marks)

(a) Evaluate $f(1)$ correct to 3 decimal places. (1 mark)

(b) Evaluate $f(0.1)$ correct to 3 decimal places. (1 mark)

(a) \[ f(1) = 2.718 \]

(b) \[ f(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6} + \ldots = 1.105 \]

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Evaluates the function for $x = 1$</td>
<td>1</td>
<td>simple</td>
</tr>
<tr>
<td>Evaluates the function for $x = 0.1$</td>
<td>1</td>
<td>simple</td>
</tr>
</tbody>
</table>
(c) Calculate the following correct to 3 decimal places and state the least number of terms needed to do so. (6 marks)

\[
\begin{align*}
  f(0.3) &= 1.345 & \text{2 terms} \\
  f(0.5) &= 1.649 & \text{5 terms} \\
  f(0.8) &= 2.226 & \text{6 terms}
\end{align*}
\]

(d) Comment on any pattern you saw from part (c). (2 marks)

As \( x \) approaches 1, we need more terms of the infinite series to approach the limiting value of the function. The values of the series converges more rapidly to its limit for small values of \( x \).

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correctly evaluates each term to 3 decimal places, terms</td>
<td>3</td>
<td>simple</td>
</tr>
<tr>
<td>Correctly states the number of terms needed</td>
<td>3</td>
<td>simple</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Refers to the limit of the function</td>
<td>1</td>
<td>simple</td>
</tr>
<tr>
<td>Refers to the rate that the function approaches the limit</td>
<td>1</td>
<td>complex</td>
</tr>
</tbody>
</table>

**Question 3** (11 marks)

(a) Evaluate the derivative of \( y = \frac{x^4}{4!} \) giving the answer in factorial form. (2 marks)

Given \( y = \frac{x^4}{4!} \) \( \Rightarrow \) \( \frac{dy}{dx} = \frac{4x^3}{4!} \)

\[
\frac{dy}{dx} = \frac{x^3}{3!}
\]

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uses the polynomial rule correctly</td>
<td>1</td>
<td>simple</td>
</tr>
<tr>
<td>Simplifies the factorial correctly</td>
<td>1</td>
<td>simple</td>
</tr>
</tbody>
</table>
(b) Hence, evaluate the derivative of the infinite series \( f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \)
giving the answer in factorial form. (3 marks)

\[
\frac{d}{dx} \left( f(x) \right) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \\
\]

\[
= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \\
\implies \frac{d}{dx} \left( f(x) \right) = f(x)
\]

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uses the polynomial rule correctly for each term</td>
<td>2</td>
<td>simple</td>
</tr>
<tr>
<td>Indicates the derivative is also an infinite series</td>
<td>1</td>
<td>complex</td>
</tr>
</tbody>
</table>

(c) Compare the result from Q3(b) above to the result in Q1(c). (1 mark)

\[ y = a^x \quad \text{and} \quad \frac{dy}{dx} = a^x \quad (\text{for a given value of } a = e) \quad \text{also} \quad y = f(x) \quad \text{and} \quad \frac{dy}{dx} = f(x) \]

This implies that for both each function is its own derivative.

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>States that each function is its own derivative</td>
<td>1</td>
<td>simple</td>
</tr>
</tbody>
</table>

(d) Compare \( f(1) \) from Q2(a) and the value for \( a \) where \( \left[ \frac{a^h - 1}{h} \right] = 1 \) from Q1(b) (1 mark)

\[
f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \\
\Rightarrow f(1) = 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!} + \ldots \approx 2.72 \quad \text{and} \quad a \approx 2.72 \\
\Rightarrow f(1) = a
\]

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shows these values are equal to 2 decimal places</td>
<td>1</td>
<td>simple</td>
</tr>
</tbody>
</table>
(e) Using \( f(1) \) correct to 3 decimal places from Q2(a) and \( h=0.0001 \), evaluate the limit

\[
\lim_{h \to 0} \frac{a^h - 1}{h}
\]

to 4 decimal places. \( \text{(1 mark)} \)

\[
\lim_{h \to 0} \frac{2.718^h - 1}{h} = 0.9999
\]

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Evaluates the limit correctly</td>
<td>1</td>
<td>simple</td>
</tr>
</tbody>
</table>

(f) Use the results from Q1–3 to make a conjecture regarding the two functions

\[
y = a^x \quad \text{and} \quad f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots
\]

Since \( f(1) = 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!} + \ldots = e \) then \( y = a^1 = e \)

Hence both functions \( y = e^x \) and \( f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \) are the same where:

\[
f(1) = e, \ f(0) = 1 \quad \text{and} \quad f(x) = f''(x)
\]

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>States the two functions are equivalent</td>
<td>1</td>
<td>complex</td>
</tr>
<tr>
<td>States each function is its own derivative</td>
<td>1</td>
<td>complex</td>
</tr>
<tr>
<td>States the limiting value for ( a = e )</td>
<td>1</td>
<td>complex</td>
</tr>
</tbody>
</table>

End of solutions
Sample assessment task
Mathematics Methods – ATAR Year 12
Test 3 – Unit 1

Assessment type: Response

Conditions:
Time for the task: Up to 50 minutes, in class, under test conditions

Materials required:
Section One: Calculator-free Standard writing equipment
Section Two: Calculator-assumed Calculator (to be provided by the student)

Other materials allowed: Drawing templates, one page of notes in Section Two

Marks available: 44
Section One: Calculator-free (23 marks)
Section Two: Calculator-assumed (21 marks)

Task weighting: 8%
Section One: Calculator-free (23 marks)

Question 1 [3.2.16] [3.2.17] (6 marks)

(a) Evaluate \( \frac{dy}{dx} \) given that: \( y = \int_{1}^{3} \sqrt{1 + t^{2}} \, dt \) (2 marks)

(b) Show that \( \int_{1}^{3} \frac{6x + 4}{\sqrt{x}} \, dx = 16\sqrt{2} - 12 \) (4 marks)

Question 2 [3.2.22] (9 marks)

A train is travelling on a straight track between two stations under the following conditions. It starts from rest at station A and moves with acceleration \( a(t) = 5 \, ms^{-2} \) for \( 0 \leq t < 4 \) seconds. It then maintains its speed for 60 seconds such that \( a(t) = 0 \, ms^{-2} \) for \( 4 \leq t < 64 \) seconds. Finally, it slows to rest at a constant rate over 10 seconds such that \( v(t) = 148 - 2t \, ms^{-1} \) for \( 64 \leq t \leq 74 \) seconds and stops in station B.

(a) Sketch the Velocity V's Time graph (5 marks)

(b) Calculate the total distance in metres between station A and station B. (4 marks)

Question 3 [3.2.19] (3 marks)

Explain why \( \int_{3}^{5} x(x - 3)(x - 5) \, dx = \int_{0}^{2} (x + 3)x(x - 2) \, dx \).
Question 4 [3.3.4]  

Below is the sample space for the tossing of two dice and recording the numbers on the upper face of each die.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1,1)</td>
<td>(1,2)</td>
<td>(1,3)</td>
<td>(1,4)</td>
<td>(1,5)</td>
<td>(1,6)</td>
</tr>
<tr>
<td>2</td>
<td>(2,1)</td>
<td>(2,2)</td>
<td>(2,3)</td>
<td>(2,4)</td>
<td>(2,5)</td>
<td>(2,6)</td>
</tr>
<tr>
<td>3</td>
<td>(3,1)</td>
<td>(3,2)</td>
<td>(3,3)</td>
<td>(3,4)</td>
<td>(3,5)</td>
<td>(3,6)</td>
</tr>
<tr>
<td>4</td>
<td>(4,1)</td>
<td>(4,2)</td>
<td>(4,3)</td>
<td>(4,4)</td>
<td>(4,5)</td>
<td>(4,6)</td>
</tr>
<tr>
<td>5</td>
<td>(5,1)</td>
<td>(5,2)</td>
<td>(5,3)</td>
<td>(5,4)</td>
<td>(5,5)</td>
<td>(5,6)</td>
</tr>
<tr>
<td>6</td>
<td>(6,1)</td>
<td>(6,2)</td>
<td>(6,3)</td>
<td>(6,4)</td>
<td>(6,5)</td>
<td>(6,6)</td>
</tr>
</tbody>
</table>

One activity is to add the numbers in each pair and record how frequently these numbers came up. For example, (3,2) gives 3+2=5.

(a) Set up a discrete probability table for the possible outcomes of this activity and give the theoretical probabilities. (2 marks)

(b) Draw a relative frequency diagram from the table. (3 marks)

End of Section One
Section Two: Calculator-assumed (21 marks)

Question 5 [3.2.18] (3 marks)

A large container has developed a leak and is losing its liquid at a rate given by the equation
\[
\frac{dv}{dt} = 3 - 3e^{0.2t} \text{ in litres per hour. Given } v = \text{volume in litres and } t = \text{time in hours, if the leak is stopped after three hours, calculate, to the nearest millilitre, how much liquid is lost in that time.}
\]

Question 6 [3.2.20] (5 marks)

(a) Evaluate the integral \[\int_{-3}^{3} (x - 3)x(x + 3)\,dx\] and explain the result. (2 marks)

(b) Evaluate the area between the graphs \(y_1 = x\) and \(y_2 = (x + 3)x(x - 3)\). (3 marks)

Question 7 [3.2.9; 3.2.21] (5 marks)

An engineer is remotely monitoring the instruments from a test car travelling in a straight line on a track. At a given instant, she noticed that the acceleration of the car was a constant 4ms\(^{-2}\) and 5 seconds later she recorded the car was travelling with a velocity of 50ms\(^{-1}\). Calculate the velocity equation of the car over this period and how far the car travelled in that time.
Question 8 [3.3.5; 3.3.6] (3 marks)

(a) In Q4 Section One of this test, you were asked to set up a discrete probability table for the possible outcomes of the two–dice activity and give the theoretical probabilities.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1,1)</td>
<td>(1,2)</td>
<td>(1,3)</td>
<td>(1,4)</td>
<td>(1,5)</td>
<td>(1,6)</td>
</tr>
<tr>
<td>2</td>
<td>(2,1)</td>
<td>(2,2)</td>
<td>(2,3)</td>
<td>(2,4)</td>
<td>(2,5)</td>
<td>(2,6)</td>
</tr>
<tr>
<td>3</td>
<td>(3,1)</td>
<td>(3,2)</td>
<td>(3,3)</td>
<td>(3,4)</td>
<td>(3,5)</td>
<td>(3,6)</td>
</tr>
<tr>
<td>4</td>
<td>(4,1)</td>
<td>(4,2)</td>
<td>(4,3)</td>
<td>(4,4)</td>
<td>(4,5)</td>
<td>(4,6)</td>
</tr>
<tr>
<td>5</td>
<td>(5,1)</td>
<td>(5,2)</td>
<td>(5,3)</td>
<td>(5,4)</td>
<td>(5,5)</td>
<td>(5,6)</td>
</tr>
<tr>
<td>6</td>
<td>(6,1)</td>
<td>(6,2)</td>
<td>(6,3)</td>
<td>(6,4)</td>
<td>(6,5)</td>
<td>(6,6)</td>
</tr>
</tbody>
</table>

Using the same table, calculate the Mean, and Standard deviation for the distribution:

(2 marks)

(b) These terms in part (a) above are referred as parameters. Explain why. (1 mark)

Question 9 [3.3.1] (5 marks)

A carton contains 12 eggs, 5 of which are brown and 7 white. A chef selects 4 eggs at random, to use in an omelette.

(a) Calculate the discrete probability distribution for \(x\) which represents the number of white eggs chosen, giving your answer in fraction form. (3 marks)

(b) Calculate the mean and standard deviation of the probability distribution. (2 marks)

\[
\begin{array}{c|cccccc}
 x & 0 & 1 & 2 & 3 & 4 \\
 \hline
 \Pr(X=x) & & & & & \\
\end{array}
\]

End of Section Two
Solutions and marking key for Test 3 – Unit 1

Section One: Calculator-free (23 marks)

Question 1 [3.2.16] [3.2.17] (6 marks)

(a) Evaluate \( \frac{dy}{dx} \) given that: \( y = \int_1^3 \sqrt{1 + t^2} \, dt \) (2 marks)

\[
y = \int_1^3 \sqrt{1 + t^2} \, dt \Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \sqrt{1 + t^2} \times 3x^2
\]

\[
= \sqrt{\left(1 + \left(x^3\right)^2\right)} \times 3x^2
\]

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculates the derivative of the integral correctly</td>
<td>1</td>
<td>complex</td>
</tr>
<tr>
<td>Applies the chain rule correctly</td>
<td>1</td>
<td>complex</td>
</tr>
</tbody>
</table>

(b) Show that \( \int_1^3 \frac{6x + 4}{\sqrt{x}} \, dx = 16\sqrt{2} - 12 \) (4 marks)

\[
\int_1^3 \frac{6x + 4}{\sqrt{x}} \, dx = \left[ \left(6x^{\frac{3}{2}} + 4x^{-\frac{1}{2}}\right) \right]_1^3
\]

\[
= \left[ 4x^{\frac{3}{2}} + 8x^{\frac{1}{2}} \right]_1^3
\]

\[
= \left(8\sqrt{3} + 8\sqrt{2}\right) - \left(4 + 8\right) = 16\sqrt{2} - 12
\]

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Partitions the algebraic fraction before integrating</td>
<td>1</td>
<td>simple</td>
</tr>
<tr>
<td>Simplifies fractional indices when dividing</td>
<td>1</td>
<td>simple</td>
</tr>
<tr>
<td>Simplifies fractions accurately when integrating</td>
<td>1</td>
<td>simple</td>
</tr>
<tr>
<td>Shows adequate working with the substitution</td>
<td>1</td>
<td>simple</td>
</tr>
</tbody>
</table>
Question 2 [3.2.21] (9 marks)

A train is travelling on a straight track between two stations under the following conditions.
It starts from rest at station A and moves with acceleration \( a(t) = 5 \, \text{ms}^{-2} \) for \( 0 \leq t < 4 \) seconds.
It then maintains its speed for 60 seconds such that \( a(t) = 0 \, \text{ms}^{-2} \) for \( 4 < t < 64 \) seconds.
Finally it slows to rest at a constant rate over 10 seconds such that
\[
v(t) = 148 - 2t \, \text{ms}^{-1}
\]
for \( 64 \leq t \leq 74 \) seconds and stops in station B.

(a) Sketch the Velocity V’s Time graph (5 marks)

\[
a(t) = 5 \, \text{ms}^{-2} \text{ for } 0 \leq t < 4 \text{ seconds } \Leftrightarrow (v) t = 5t + c
\]
\[
v(0) = 0 \Leftrightarrow c = 0 : (v) t = 5t \text{ for } 0 \leq t < 4 \text{ seconds}
\]
t \( \rightarrow 4 \) then \( v \rightarrow 20 \) \( \therefore v = 20 \, \text{ms}^{-1} \) for \( 4 \leq t < 64 \) seconds.

Also \( v(t) = 148 - 2t \, \text{ms}^{-1} \) for \( 64 \leq t \leq 74 \) seconds.

(b) Calculate the total distance in metres between station A and station B. (4 marks)

Distance travelled = \( \int v(t) \, dt \)
\[
= \int_0^4 5t \, dt + \int_{4}^{64} 20 \, dt + \int_{64}^{74} 148 - 2t \, dt
\]
\[
= 40 + 1200 + 100 = 1340 \, \text{m}
\]

Specific behaviours | Mark allocation | Item classification
--- | --- | ---
Determines the first two velocity functions | 2 | simple
Draws each section of the graph accurately | 3 | simple

Specific behaviours | Mark allocation | Item classification
--- | --- | ---
Uses the velocity functions/graphs to calculate the distance travelled for each leg | 3 | simple
States the correct distance travelled | 1 | simple
Question 3 [3.2.19] (3 marks)

Explain why \( \int_{2}^{3} x(x - 2)(x - 5)\,dx = \int_{0}^{1}(x + 2)x(x - 3)\,dx \)

The original graph has been translated two units to the left and the limits for the integral have also been translated two units to the left.
Hence, the area to be calculated in both cases is the same area enclosed by the function below the x-axis.

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>States the graph has been translated to the left</td>
<td>1</td>
<td>complex</td>
</tr>
<tr>
<td>States the limits have also been translated two units left</td>
<td>1</td>
<td>complex</td>
</tr>
<tr>
<td>States the area is the same in both cases</td>
<td>1</td>
<td>complex</td>
</tr>
</tbody>
</table>

Question 4 [3.3.4] (5 marks)

(a) Set up a discrete probability table for the possible outcomes of this activity and give the theoretical probabilities.

<table>
<thead>
<tr>
<th>x</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pr(x)</td>
<td>( \frac{1}{36} )</td>
<td>( \frac{2}{36} )</td>
<td>( \frac{3}{36} )</td>
<td>( \frac{4}{36} )</td>
<td>( \frac{5}{36} )</td>
<td>( \frac{6}{36} )</td>
<td>( \frac{5}{36} )</td>
<td>( \frac{4}{36} )</td>
<td>( \frac{3}{36} )</td>
<td>( \frac{2}{36} )</td>
<td>( \frac{1}{36} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Defines the set of variables correctly</td>
<td>1</td>
<td>simple</td>
</tr>
<tr>
<td>Completes the probability values</td>
<td>1</td>
<td>simple</td>
</tr>
</tbody>
</table>
(b) Draw a relative frequency diagram from the table. (3 marks)

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Centres each class on 2, 3 ... etc.</td>
<td>1</td>
<td>simple</td>
</tr>
<tr>
<td>Sets an appropriate horizontal scale</td>
<td>1</td>
<td>simple</td>
</tr>
<tr>
<td>Draws a good representation of the histogram</td>
<td>1</td>
<td>simple</td>
</tr>
</tbody>
</table>
Solutions and marking key for Test 3 – Unit 1

Section Two: Calculator-assumed (21 marks)

Question 5 [3.2.18] (3 marks)

A large container has developed a leak and is losing its liquid at a rate given by the equation \( \frac{dv}{dt} = 3 \cdot 3e^{-0.2t} \) in litres per hour. Given \( v = \) volume in litres and \( t = \) time in hours, if the leak is stopped after three hours, calculate, to the nearest millilitre, how much liquid is lost in that time.

If \( \frac{dv}{dt} = 3 \cdot 3e^{-0.2t} \) then liquid lost in three hours

\[
\int_{0}^{3} 3 \cdot 3e^{-0.2t} \, dt = 2.232 \text{ litres}
\]

\( = 2232 \text{ ml} \)

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sets up the correct integral</td>
<td>1</td>
<td>simple</td>
</tr>
<tr>
<td>Sets up the correct limits</td>
<td>1</td>
<td>simple</td>
</tr>
<tr>
<td>States the correct volume to the nearest millilitre</td>
<td>1</td>
<td>simple</td>
</tr>
</tbody>
</table>
Question 6 [3.2.20] 

(a) Evaluate the integral \( \int_{-3}^{3} (x - 3)x(x + 3)\,dx \) and explain the result. 

\[ \int_{-3}^{3} (x - 3)x(x + 3)\,dx = 0 \]

Since the graph is symmetrical about the origin the area above the x-axis (+) equals the area below the x-axis (−). Hence these areas add to zero.

(b) Evaluate the area between the graphs \( y_1 = x \) and \( y_2 = (x + 3)x(x - 3) \)

Required area
\[
= \int_{-3.162}^{3.162} |y_2 - y_1|\,dx \\
= 2 \times \int_{-3.162}^{0} x(x - 3)(x + 3) - x\,dx \\
= 50 \text{ sq units}
\]
Question 7 [3.2.9; 3.2.21] (5 marks)

An engineer is remotely monitoring the instruments from a test car travelling in a straight line on a track. At a given instant she noticed that the acceleration of the car was a constant 4 ms$^{-2}$ and 5 seconds later she recorded the car was travelling with a velocity of 50 ms$^{-1}$. Calculate the velocity equation of the car over this period and how far the car travelled in that time.

Given $a = 4\text{ms}^{-2}$ and $v = \int a \, dt$ where $a$ is a constant

\[ v = 4t + c \text{ since } v = 50 \text{ when } t = 5 \]
\[ 50 = 20 + c \Rightarrow c = 30 \]
\[ v(t) = 4t + 30 \]

∴ Distance travelled \[ \int_{0}^{5} (4t + 30) \, dt = 200 \text{m} \]

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculates the correct constant of integration</td>
<td>1</td>
<td>simple</td>
</tr>
<tr>
<td>Gives the correct velocity equation</td>
<td>1</td>
<td>simple</td>
</tr>
<tr>
<td>Uses the integral of the velocity equation to calculate the distance travelled</td>
<td>1</td>
<td>simple</td>
</tr>
<tr>
<td>Uses the correct limits</td>
<td>1</td>
<td>simple</td>
</tr>
<tr>
<td>Calculates the correct distance</td>
<td>1</td>
<td>simple</td>
</tr>
</tbody>
</table>
Question 8 [3.3.5; 3.3.6] (3 marks)

(a) In Q4 Section One, you were asked set up a discrete probability table for the possible outcomes of the two-dice activity and give the theoretical probabilities.

Using the same table, calculate the Mean, and Standard deviation for the distribution:

\[
\text{Mean} = 7 \\
\text{Standard deviation} = 2.4152
\]

(b) These terms in part (a) above are referred to as parameters. Explain why. (1 mark)

Parameters refer to the measures of a population or a theoretical probability distribution.
Question 9 [3.3.1] (5 marks)

A carton contains 12 eggs, 5 of which are brown and 7 white. A chef selects 4 eggs at random, to use in an omelette.

(a) Calculate the discrete probability distribution for $x$ which represents the number of white eggs chosen, giving your answer in fraction form. (3 marks)

(b) Calculate the mean and standard deviation of the probability distribution. (2 marks)

<table>
<thead>
<tr>
<th>$x$</th>
<th>Pr($X=x$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/99</td>
</tr>
<tr>
<td>1</td>
<td>14/99</td>
</tr>
<tr>
<td>2</td>
<td>42/99</td>
</tr>
<tr>
<td>3</td>
<td>35/99</td>
</tr>
<tr>
<td>4</td>
<td>7/99</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Specific behaviours</th>
<th>Mark allocation</th>
<th>Item classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shows appropriate working for at least one value</td>
<td>1</td>
<td>simple</td>
</tr>
<tr>
<td>Calculates the five values accurately</td>
<td>2</td>
<td>simple</td>
</tr>
<tr>
<td>Gives the correct mean</td>
<td>1</td>
<td>simple</td>
</tr>
<tr>
<td>Gives the correct standard deviation</td>
<td>1</td>
<td>simple</td>
</tr>
</tbody>
</table>

Mean = 2.3333
Standard deviation = 0.8409

End of solutions