Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS SPECIALIST

Calculator-free

ATAR course examination 2016

Marking Key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2016/64828

Section One: Calculator-free **35% (53 marks)**

Question 1 **(4 marks)**

Functions f and g are defined as $f(x) = \ln(x)$ and $g(x) = \dfrac{1}{x}$.

(a) Determine an expression for $g \circ f(x)$. (1 mark)

Solution
$g \circ f(x) = g(\ln(x)) = \dfrac{1}{\ln(x)}$
Specific behaviours
✓ writes the correct expression for $g \circ f(x)$

(b) For $g \circ f(x)$, state the:

(i) domain. (2 marks)

Solution
$D_{gof} = \{x : x > 0,\ x \neq 1\}$
Specific behaviours
✓ states $x > 0$ ✓ states $x \neq 1$

(ii) range. (1 mark)

Solution
$R_{gof} = \{y : y \neq 0\}$
Specific behaviours
✓ states $y \neq 0$

Question 2 **(7 marks)**

Give exact expressions for each of the following in the form $a+bi$:

(a) $\dfrac{\overline{2+i}}{(1-i)^2}$. (3 marks)

Solution
$\dfrac{\overline{2+i}}{(1-i)^2} = \dfrac{2-i}{-2i} \times \dfrac{i}{i}$
$= \dfrac{2i-i^2}{-2(-1)} = \dfrac{2i+1}{2} = \dfrac{1}{2}+i$
Specific behaviours
✓ writes the conjugate and expands the denominator correctly ✓ multiplies by a form of one correctly to determine a real denominator ✓ simplifies correctly in the form $a+bi$

(b) $\left(\sqrt{3}-i\right)^5$. (4 marks)

Solution
$\left(\sqrt{3}-i\right)^5 = \left(2cis\left(-\dfrac{\pi}{6}\right)\right)^5$
$= 32cis\left(-\dfrac{5\pi}{6}\right)$
$= 32\left(\cos\left(-\dfrac{5\pi}{6}\right)+i\sin\left(-\dfrac{5\pi}{6}\right)\right)$
$= 32\left(-\dfrac{\sqrt{3}}{2}+i\left(-\dfrac{1}{2}\right)\right)$
$= -16\sqrt{3}-16i$
Specific behaviours
✓ determines the modulus of the polar form correctly ✓ determines the argument of the polar form correctly ✓ applies DeMoivre's Theorem correctly ✓ simplifies correctly in the form $a+bi$

Alternative solution for Question Q2(b).

Alternative Solution
$\left(\sqrt{3}-i\right)^5$ $= \left(\sqrt{3}\right)^5 + 5\left(\sqrt{3}\right)^4(-i) + 10\left(\sqrt{3}\right)^3(-i)^2 + 10\left(\sqrt{3}\right)^2(-i)^3 + 5\left(\sqrt{3}\right)(-i)^4 + (-i)^5$ $= 9\sqrt{3} + 45(-i) + 30\sqrt{3}(-1) + 30(-i)(-1) + 5\sqrt{3}(1) + (1)(-i)$ $= 9\sqrt{3} - 45i - 30\sqrt{3} + 30i + 5\sqrt{3} - i$ $= -16\sqrt{3}-16i$
Specific behaviours
✓ expands using the binomial theorem correctly   ✓ evaluates powers of $\sqrt{3}$ correctly   ✓ evaluates powers of i correctly   ✓ simplifies correctly in the form $a+bi$

Question 3 **(5 marks)**

Consider $f(z) = z^3 + 2z^2 - 5z + 12$ where z is a complex number.

(a) Show that $(z+4)$ is a factor of $f(z)$. (2 marks)

Solution
$f(-4) = -64 + 32 + 20 + 12 = 0$ $\therefore$ $(z+4)$ is a factor of $f(z)$
Specific behaviours
✓ substitutes $z = -4$ correctly ✓ provides evidence that i.e. not just the statement $f(-4) = 0$

(b) Solve the equation $z^3 + 2z^2 - 5z + 12 = 0$. (3 marks)

Solution
$f(z) = (z+4)(z^2 - 2z + 3) = 0$ i.e. $(z+4)\left((z-1)^2 + 2\right) = 0$ $\therefore$ $z = -4$ or $(z-1)^2 = -2$ $\therefore$ $z = -4$ or $z = 1 \pm \sqrt{2}i$
Specific behaviours
✓ determines the quadratic factor correctly ✓ states that $z = -4$ is a solution ✓ determines the complex solutions to the quadratic equation correctly

Question 4 **(6 marks)**

(a) Express $\dfrac{x-8}{(x+2)(x-3)}$ in the form $\dfrac{a}{x+2}+\dfrac{b}{x-3}$. (3 marks)

Solution
$\dfrac{a}{x+2}+\dfrac{b}{x-3} = \dfrac{a(x-3)+b(x+2)}{(x+2)(x-3)}$ $= \dfrac{(a+b)x+(2b-3a)}{(x+2)(x-3)} = \dfrac{x-8}{(x+2)(x-3)}$ Hence $\left.\begin{aligned} a+b&=1 \\ 2b-3a&=-8 \end{aligned}\right\}$ solving gives $a=2,\ b=-1$ i.e. $\dfrac{x-8}{(x+2)(x-3)} = \dfrac{2}{x+2} - \dfrac{1}{x-3}$
Specific behaviours
✓ obtains the correct expression for the equivalent numerator in terms of a, b, x ✓ forms the equations to solve for a, b correctly   ✓ determines the correct values for a, b

(b) Hence determine $\displaystyle\int \frac{x-8}{(x+2)(x-3)}\,dx$. (3 marks)

Solution
$\displaystyle\int \frac{x-8}{(x+2)(x-3)}\,dx = \int\left(\frac{2}{x+2} - \frac{1}{x-3}\right)dx$ $= 2\ln\lvert x+2\rvert - \ln\lvert x-3\rvert + c$ $= \ln\left\lvert\dfrac{(x+2)^2}{(x-3)}\right\rvert + c$
Specific behaviours
✓ substitutes for the integrand using the result from part (a)   ✓ anti-differentiates correctly using the natural logarithm function   ✓ uses a constant of integration with the anti-derivative

Question 5 **(7 marks)**

Evaluate the following definite integrals exactly.

(a) $\int_0^{\frac{\pi}{4}} 12\sin^4 2x \cos 2x \; dx$ Put $u = \sin 2x$ (4 marks)

Solution
$\int_0^{\frac{\pi}{4}} 12\sin^4 2x\cos 2x \; dx = \int_0^1 12u^4.\cos 2x.\frac{du}{2\cos 2x} = \int_0^1 6u^4 du$
$= \left[\frac{6u^5}{5}\right]_0^1 = \left(\frac{6(1^5)}{5} - 0\right) = \frac{6}{5}$
Specific behaviours
✓ expresses the integrand correctly in terms of u ✓ changes the limits of integration correctly ✓ writes the correct anti-derivative ✓ evaluates correctly

(b) $\int_0^{\frac{1}{2}} \tan^2\left(\frac{\pi x}{2}\right) dx$ (3 marks)

Solution
$\int_0^{\frac{1}{2}} \tan^2\left(\frac{\pi x}{2}\right) dx = \int_0^{\frac{1}{2}} \left(\sec^2\left(\frac{\pi x}{2}\right) - 1\right) dx$
$= \left[\frac{2}{\pi}\tan\left(\frac{\pi x}{2}\right) - x\right]_0^{\frac{1}{2}}$
$= \left(\frac{2}{\pi}\tan\left(\frac{\pi}{4}\right) - \frac{1}{2}\right) - \left(\frac{2}{\pi}\tan(0) - 0\right) = \frac{2}{\pi} - \frac{1}{2}$
Specific behaviours
✓ uses the trigonometric identity to express the integrand in terms of $\sec^2\left(\frac{\pi x}{2}\right)$ ✓ anti-differentiates correctly ✓ evaluates correctly

Question 6 **(6 marks)**

(a) Solve the system of equations. (3 marks)

$$x + y + z = 4 \quad \ldots (1)$$
$$3x - y + z = 8 \quad \ldots (2)$$
$$2x - y + z = 0 \quad \ldots (3)$$

Solution
Consider $(2)-(3) \quad \therefore \quad x = 8$ $(1): \quad y + z = -4$ $(2): \quad -y + z = -16$ $(1)+(2): \quad 2z = -20$ $\therefore \quad z = -10$ $\therefore \quad y = 6$ Hence the solution is $x = 8$ $y = 6$ $z = -10$
Specific behaviours
✓ eliminates a variable correctly using an appropriate technique ✓ solves correctly for the first variable ✓ solves correctly for the second and third variables

Suppose that the third equation in part (a) is changed to $2x - y + kz = 0$. The first two equations remain unchanged.

(b) Determine the value of the constant k so that the changed system of equations has no solution. (3 marks)

Solution
System is now : $x + y + z = 4 \quad \ldots (1)$ $3x - y + z = 8 \quad \ldots (2)$ $2x - y + kz = 0 \quad \ldots (3)$ Consider $(1)+(2)$: $4x + 2z = 12 \quad \ldots(4)$ $(2)-(3)$: $x + (1-k)z = 8 \quad \ldots(5)$ From (5): $x = 8-(1-k)z$ substituting into (4): $4\left(8-(1-k)z\right)+2z=12$ i.e. $4kz - 2z = -20$ i.e. $z(4k-2) = -20$ Hence for there to be no solution we require $4k - 2 = 0$ i.e. $k = \frac{1}{2}$
Specific behaviours
✓ eliminates two variables to express z correctly (or another variable) in terms of k ✓ states that the variable coefficient must be ZERO for no solution ✓ determines the value of k

Question 7 **(7 marks)**

Points A, B have respective position vectors $\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix}$.

(a) Determine the vector equation for the sphere that has $\overline{AB}$ as its diameter. (3 marks)

Solution
Centre point $C = \frac{1}{2}\left(\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ Radius $r = \lvert\overrightarrow{AC}\rvert = \left\lvert \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\rvert = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$ Equation for circle with diameter $\overline{AB}$: $\left\lvert \underset{\sim}{r} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \right\rvert = \sqrt{6}$
Specific behaviours
✓ determines the position vector for the centre correctly ✓ determines the radius correctly ✓ forms the vector equation for the sphere correctly

If point O is the origin, consider the plane that contains the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

(b) Determine the vector equation for this plane in the form $\underset{\sim}{r} \cdot \underset{\sim}{n} = c$. (4 marks)

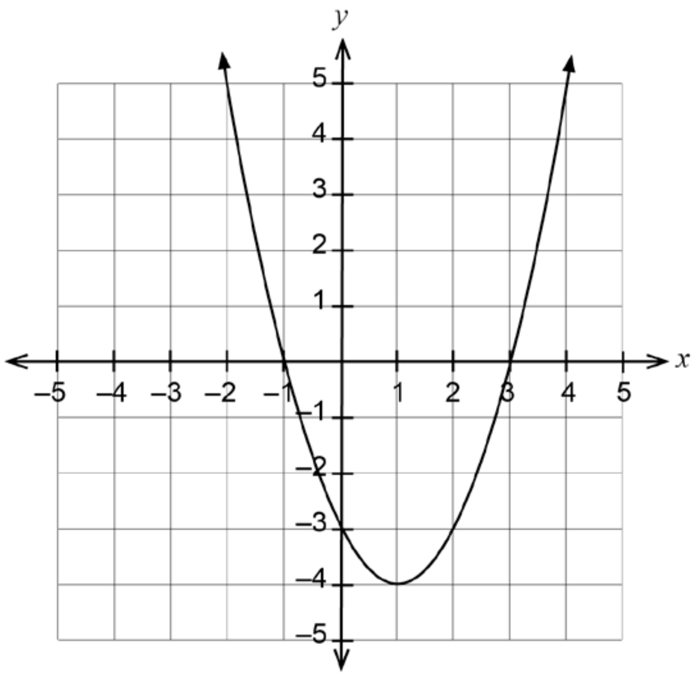
Solution
Use $\underset{\sim}{n} = \overrightarrow{OA} \times \overrightarrow{OB} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} \times \begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 0(5)-(-2)(3) \\ 0(3)-4(5) \\ 4(-2)-0(0) \end{pmatrix} = \begin{pmatrix} 6 \\ -20 \\ -8 \end{pmatrix}$ or $\begin{pmatrix} 3k \\ -10k \\ -4k \end{pmatrix}$
Since $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in$ plane, then $c = 0$ i.e. equation of plane is $\underset{\sim}{r} \cdot \begin{pmatrix} 6 \\ -20 \\ -8 \end{pmatrix} = 0$
Specific behaviours
✓ uses the idea of the cross product of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ to determine the normal ✓ determines the cross product correctly ✓ states that the constant $c = 0$ ✓ forms the vector equation for the plane correctly

or

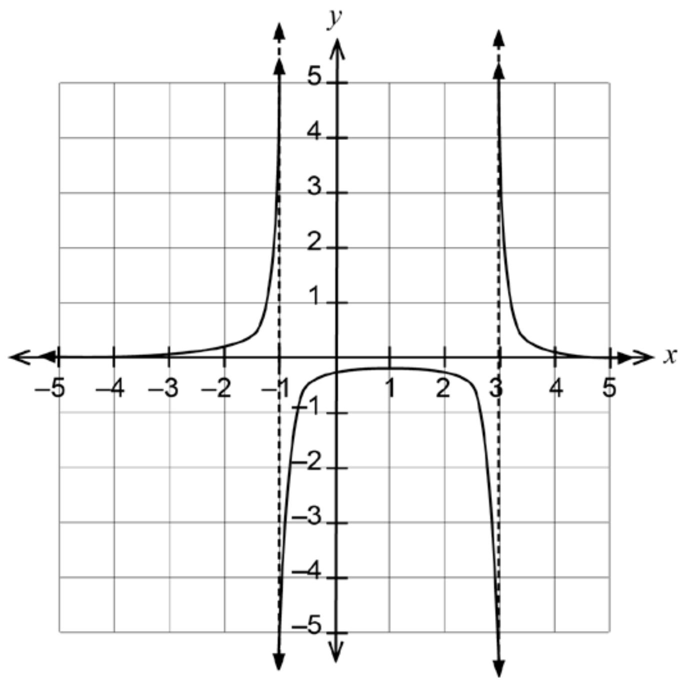
Alternative Solution
Let $\underset{\sim}{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ Hence $\overrightarrow{OA} \cdot \underset{\sim}{n} = 0$ i.e. $4a + 3c = 0$
Also $\overrightarrow{OB} \cdot \underset{\sim}{n} = 0$ i.e. $-2b + 5c = 0$
Choose $a = -3,\ b = 10,\ c = -4$
Since $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in$ Plane, then $c = 0$ i.e. equation of plane is $\underset{\sim}{r} \cdot \begin{pmatrix} 6 \\ -20 \\ -8 \end{pmatrix} = 0$
Specific behaviours
✓ uses the idea that the dot product with the normal vector must be ZERO ✓ determines the normal vector from the dot product equations ✓ states that the constant $c = 0$ ✓ forms the vector equation for the plane correctly

Question 8 **(11 marks)**

The graph of $f(x) = (x-1)^2 - 4$ is shown below.

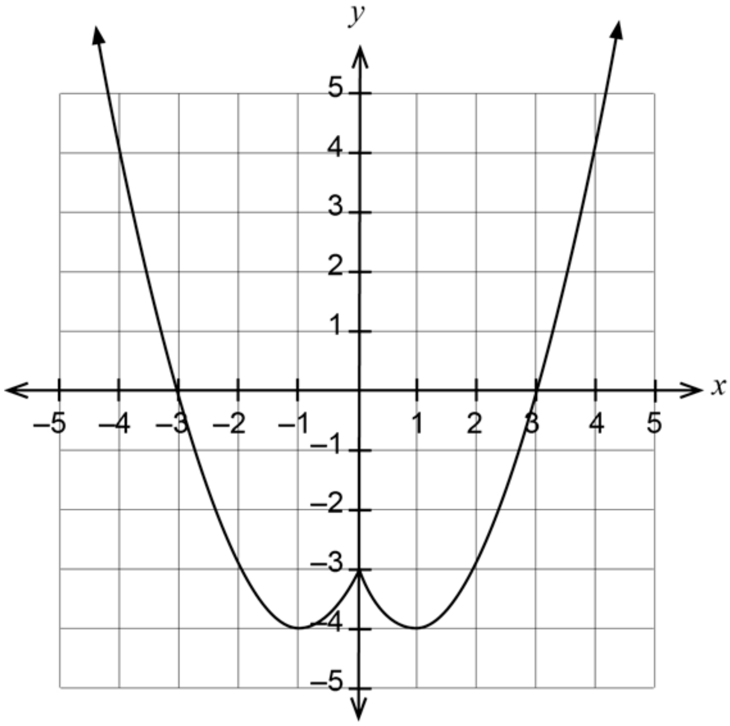


(a) Sketch the graph of $y = \dfrac{1}{f(x)}$ on the coordinate axes below. (4 marks)



Solution
Indicated on the graph above.
Specific behaviours
✓ indicates vertical asymptotes at $x = -1$ and $x = 3$
✓ indicates $y \to 0^+$ for $\lvert x \rvert \to \infty$
✓ indicates a local maximum at $x = 1$
✓ indicates the correct curvature and behaviour around $x = -1$ and $x = 3$

(b) Sketch the graph of $y = f(|x|)$ on the coordinate axes below. (2 marks)



Solution
Indicated on the graph above.
Specific behaviours
✓ indicates the point $(-3, 0)$ on the graph ✓ indicates symmetry about $x = 0$

(c) The domain of function f is restricted to $x \le k$ so that $y = f^{-1}(x)$ is a function. If this restricted domain represents the largest possible domain, state the value for the constant k. Explain. (2 marks)

Solution
Restrict the domain of f to $\{x \mid x \le 1\}$ i.e. $k = 1$ This is chosen so that function f is a one-to-one function OR function f will be strictly decreasing (or stationary) and not decreasing and then increasing.
Specific behaviours
✓ states the correct domain or states the value for k ✓ provides an adequate explanation that f will be one-to-one

(d) Using the restriction $x \le k$, determine the defining rule for $y = f^{-1}(x)$.
Also state the domain for $y = f^{-1}(x)$. (3 marks)

Solution
$f: \ y=(x-1)^2-4 \quad \therefore \ f^{-1}: \ x=(y-1)^2-4$ i.e. $x+4=(y-1)^2$ i.e. $y-1=-\sqrt{x+4}$ since $R_{f^{-1}}=D_f$ $\therefore \ f^{-1}(x) = 1-\sqrt{x+4}$, $x \ge -4$ since $R_f = D_{f^{-1}}$
Specific behaviours
✓ interchanges x, y to write the rule for the inverse ✓ obtains the correct defining rule for $y = f^{-1}(x)$ ✓ states the correct domain for $y = f^{-1}(x)$

End of questions

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