Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS METHODS

Calculator-free

ATAR course examination 2018

Ratified Marking Key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2018/62747

Section One: Calculator-free **35% (52 Marks)**

Question 1 **(9 marks)**

A bag contains 1 red marble and 4 green marbles. A single marble is drawn from the bag. The random variable Y is defined as the number of green marbles drawn from the bag.

(a) Complete the probability distribution for Y shown below. (2 marks)

y	0	1
$P(Y = y)$	$\frac{1}{5}$	$\frac{4}{5}$

Solution
See table
Specific behaviours
✓ completes first probability correctly ✓ completes second probability correctly

(b) State the distribution of Y. (1 marks)

Solution
It is a Bernoulli distribution.
Specific behaviours
✓ states the distribution name

(c) Determine the mean and standard deviation of the distribution. (2 marks)

Solution
$\mu = \frac{4}{5}$ $\sigma = \sqrt{\frac{1}{5} \times \frac{4}{5}} = \frac{2}{5}$
Specific behaviours
✓ states the mean ✓ states the simplified value of the standard deviation

The above process is repeated five times, with the marble being replaced every time. The random variable X is defined as the number of green marbles drawn from the bag in five attempts.

(d) State the distribution of X, including its parameters. (2 marks)

Solution
$X \sim BIN\left(5, \frac{4}{5}\right)$
Specific behaviours
✓ states the distribution name ✓ states the parameters of the distribution

(e) Evaluate the probability of selecting exactly two green marbles. (2 marks)

Solution
$P\left(X=2\right)=\binom{5}{2}\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^3$ $=\frac{5\times4\times16}{2\times5^5}$ $=\frac{32}{625}$
Specific behaviours
✓ correctly substitutes into binomial formula ✓ states simplified probability

Question 2 **(6 marks)**

For a set of data values that are normally distributed, approximately 68% of the values will lie within one standard deviation of the mean, approximately 95% of the values will lie within two standard deviations of the mean and approximately 99.7% of the values will lie within three standard deviations of the mean.

If the heights of a large group of women are normally distributed with a mean μ = 163 cm and standard deviation σ = 7 cm, use the above information to answer the following questions:

(a) A statistician says that almost all of the women have heights in the range 142 cm to 184 cm. Comment on her statement. (2 marks)

Solution
Her comment is appropriate as the range corresponds to 3 standard deviations above and below the mean, which equates to approximately 99.7% of the group.
Specific behaviours
✓ states that the comment is appropriate ✓ refers to the standard deviation and 99.7%

(b) Approximately what percentage of women in the group has a height greater than 170 cm? (2 marks)

Solution
$170-163=7 \Rightarrow 1 \text{ SD above}$ $\text{Percentage} = \dfrac{100-68}{2}$ $=16$
Specific behaviours
✓ states 1 standard deviation above ✓ determines correct percentage

(c) Approximately 2.5% of the women are shorter than what height? (2 marks)

Solution
$\text{percentage} = 100-2\times 2.5$ $=95\%$ $2 \text{ SDs below} = 163-14$ $=149 \text{ cm}$
Specific behaviours
✓ determines 95% ✓ states height

Question 3 **(12 marks)**

(a) Differentiate $\left(2x^3+1\right)^5$. (2 marks)

Solution
$\frac{d}{dx}\left(2x^3+1\right)^5 = 5\times\left(6x^2\right)\left(2x^3+1\right)^4$ $= 30x^2\left(2x^3+1\right)^4$
Specific behaviours
✓ demonstrates use of the chain rule by including the $\left(2x^3+1\right)^4$ term ✓ fully determines derivative correctly

(b) Given $g'(x) = e^{2x}\sin(3x)$, determine a simplified value for the rate of change of $g'(x)$ when $x = \frac{\pi}{2}$. (3 marks)

Solution
$g''(x) = 2e^{2x}\sin(3x) + 3e^{2x}\cos(3x)$ $g''(\frac{\pi}{2}) = 2e^{\pi}\sin(\frac{3\pi}{2}) + 3e^{\pi}\cos(\frac{3\pi}{2})$ $= 2e^{\pi}(-1) + 3e^{\pi}(0)$ $= -2e^{\pi}$
Specific behaviours
✓ demonstrates the use of the product rule by stating an expression involving two terms with one term correct ✓ determines both terms of the expression correctly ✓ substitutes and determines the simplified value

Question 3 (continued)

(c) Determine the following:

(i) $\int 3\cos(2x)\,dx$. (2 marks)

Solution
$\int 3\cos(2x)\,dx = \frac{3}{2}\sin(2x) + C$
Specific behaviours
✓ determines integral including $\sin(2x)$ ✓ determines integral fully correct including constant

(ii) $\int_0^1 \frac{3x+1}{3x^2+2x+1}\,dx$. (3 marks)

Solution
$\int_0^1 \frac{3x+1}{3x^2+2x+1}\,dx = \frac{1}{2}\int_0^1 \frac{6x+2}{3x^2+2x+1}\,dx$ $= \frac{1}{2}\left[\ln\left(3x^2+2x+1\right)\right]_0^1$ $= \frac{1}{2}\left[\ln 6 - \ln 1\right]$ $= \frac{1}{2}\ln 6$
Specific behaviours
✓ modifies the integrand so the numerator function is the derivative of the denominator function ✓ correctly determines an expression for the integral ✓ substitutes and uses log laws to determine a simplified answer

(d) If $f'(x) = e^{-2x}$, find the expression for $y = f(x)$, given $f(0) = -2$. (2 marks)

Solution
$y = \int f'(x) = \frac{-e^{-2x}}{2} + c$ $at\ \ x = 0\ \ y = -2 \quad \therefore\ c = \frac{-3}{2}$ $y = \frac{-e^{-2x}}{2} - \frac{3}{2}$
Specific behaviours
✓ correctly integrates *f '(x)* and includes a constant ✓ determines the correct value of the constant

Question 4 **(4 marks)**

Ten shop owners in a coastal resort were asked how many extra staff they intended to hire for the next holiday *season. Their responses are shown below:*

3, 0, 2, 1, 2, 1, 1, 0, 2, 1

If N = number of additional staff,

(a) complete the probability distribution of N below. (2 marks)

n	0	1	2	3
P ($N = n$)	$\mathbf{\frac{2}{10}}$	$\mathbf{\frac{4}{10}}$	$\mathbf{\frac{3}{10}}$	$\mathbf{\frac{1}{10}}$

Solution
See table
Specific behaviours
✓ gives one correct entry ✓ completes the table correctly

(b) what is the mean number of staff the shop owners intend to hire? (2 marks)

Solution
$E(X) = \left(0\times\frac{2}{10}\right)+\left(1\times\frac{4}{10}\right)+\left(2\times\frac{3}{10}\right)+\left(3\times\frac{1}{10}\right)$ $=\frac{13}{10} = 1.3$
Specific behaviours
✓ gives correct expression for $E(X)$ ✓ simplifies answer

Question 5 **(3 marks)**

A 95% confidence interval for a population proportion based on a sample size of 200 has width w. What sample size is required to obtain a 95% confidence interval of width $\frac{w}{3}$?

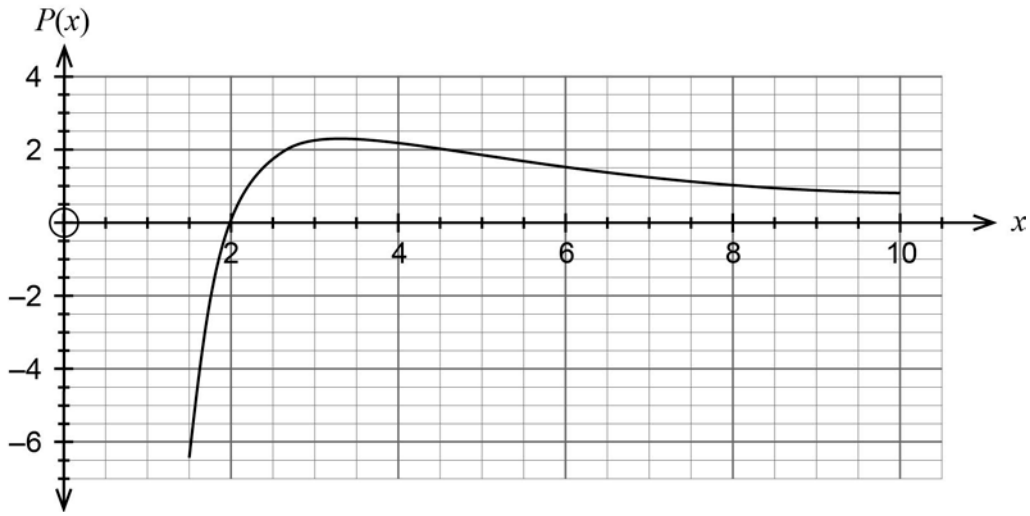
Solution
The width of a confidence interval is inversely proportional to the square root of sample size. Therefore, to have one third the width of the confidence interval requires a sample size nine times as large, so a sample size of 1800 is needed. **or** $$w = \frac{z\sigma}{\sqrt{n}}$$ (1 mark) sample size $n_1: \frac{w}{3} = \frac{z\sigma}{\sqrt{n_1}}$ (2 marks) dividing (1): $\frac{w}{3} = \frac{z\sigma}{3\sqrt{n}}$ substituting for $\frac{w}{3}$ into (2): $\frac{z\sigma}{3\sqrt{n}} = \frac{z\sigma}{\sqrt{n_1}}$ solving for $n_1: n_1 = 9n = 1800$.
Specific behaviours
✓ uses the new width as $\frac{w}{3}$ ✓ states that the sample size is nine times as large ✓ gives correct value of sample size **or** ✓ obtains equation (1) and (2) ✓ solves equation ✓ obtains correct sample size

Question 6 **(8 marks)**

A company manufactures and sells an item for $\$x$. The profit, $\$P$, made by the company per item sold is dependent on the selling price and can be modelled by the function:

$$P(x) = \frac{50\ln\left(\frac{x}{2}\right)}{x^2} \quad \text{where } 1.5 \le x \le 10\,.$$

The graph of $P(x)$ is shown below:



(a) Describe how the profit per item sold varies as the selling price changes. (3 marks)

Solution
The company will make a loss for a selling price between \$1.50 and \$2.00. The profit then increases up to approximately \$2.25 per item sold for a selling price of approximately \$3.25, and then decreases steadily to a value of less than \$1 per item sold for a selling price of \$10.
Specific behaviours
✓ states initially making a loss ✓ states profit increases to maximum at \$3.25 ✓ states it decreases after that

Question 6 (continued)

(b) Determine the exact price that should be charged for the item if the company wishes to maximise the profit per item sold. (5 marks)

Solution
$\frac{dP}{dx}=\frac{x^2\left(\frac{50}{2}\times\frac{2}{x}\right)-2x\times 50\ln\left(\frac{x}{2}\right)}{x^4}$ $=\frac{50x-100x\ln\left(\frac{x}{2}\right)}{x^4}$ $=\frac{50-100\ln\left(\frac{x}{2}\right)}{x^3}$ For max, $\frac{dP}{dx}=0\Rightarrow 0=\frac{50-100\ln\left(\frac{x}{2}\right)}{x^3}$ $\ln\left(\frac{x}{2}\right)=\frac{1}{2}$ $x=2e^{\frac{1}{2}}$
Specific behaviours
✓ correctly states the numerator of the quotient rule   ✓ correctly states the denominator of the quotient rule   ✓ simplifies derivative   ✓ equates simplified derivative to zero   ✓ determines exact value of *x*

Question 7 **(10 marks)**

(a) Determine a simplified expression for $\frac{d}{dx}\left(x\ln(x)\right)$. (2 marks)

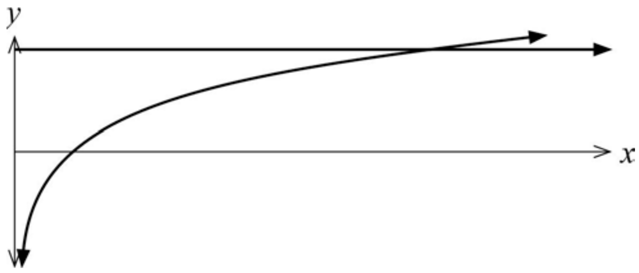
Solution
$\frac{d}{dx}\left(x\ln(x)\right) = x \times \frac{1}{x} + \ln\left(x\right)$ $= 1 + \ln\left(x\right)$
Specific behaviours
✓ uses product rule to determine derivative   ✓ simplifies the derivative

(b) Use your answer from part (a) to show that $\int \ln(x)dx = x\ln(x) - x + c$, where c is a constant. (4 marks)

Solution
$\frac{d}{dx}\left(x\ln(x)\right) = 1 + \ln\left(x\right)$ $\int \frac{d}{dx}\left(x\ln(x)\right)\,dx = \int\left(1 + \ln\left(x\right)\right)\,dx$ $x\ln(x) = x + \int \ln\left(x\right)\,dx + c$ $\int \ln\left(x\right)\,dx = x\ln(x) - x + c$
Specific behaviours
✓ integrates both sides of answer from part (a)   ✓ partly integrates right-hand side to get x ✓ uses fundamental theorem of calculus to simplify the left-hand side   ✓ rearranges to give the required result

Question 7 (continued)

The graphs of the functions $f(x)=5$ and $g(x)=\ln(x)$ are shown below.



(c) Determine the exact area enclosed between the *x*-axis, the *y*-axis and the functions $f(x)$ and $g(x)$. (4 marks)

Solution
Intersect when: $\ln(x)=5 \Rightarrow x=e^5$ Area under $f(x)$: $\int_1^{e^5} \ln(x)dx = \left[x\ln(x)-x\right]_1^{e^5}$ $=5e^5-e^5+1$ Required area $=5\times e^5-\left(5e^5-e^5+1\right)$ $=e^5-1$
Specific behaviours
✓ determines point of intersection between $f(x)$ and $g(x)$ ✓ states an integral for the area under $f(x)$ ✓ evaluates integral ✓ determines required area

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