Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS SPECIALIST

Calculator-assumed

ATAR course examination 2020

Marking key

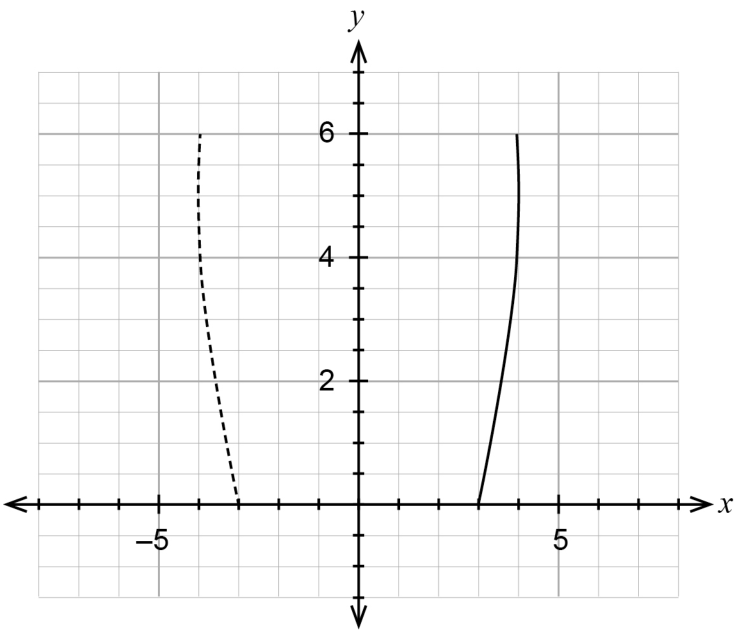
Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2020/66025

Section Two: Calculator-assumed **65% (86 Marks)**

Question 9 **(4 marks)**

The shape of a small wine glass is modelled by revolving the curve $\sin\left(\frac{y}{\pi}\right) = x - 3$ about the y axis, where $0 \le y \le 6$. All dimensions are in centimetres.

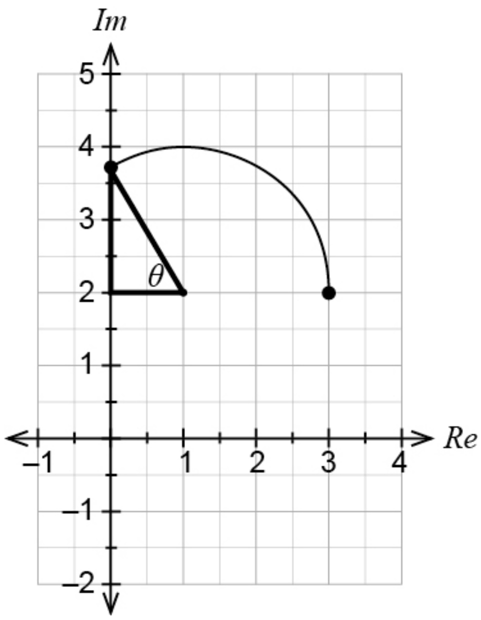


Calculate, correct to the nearest 0.01 cm, the depth of wine in the glass if it is to contain 80% of its maximum volume.

Solution
Volume when full $= \int_0^6 \pi\left(\sin\left(\frac{y}{\pi}\right)+3\right)^2 dy = 259.53228\ldots$ cm^3 Let h = the depth for 80% volume Require $\int_0^h \pi\left(\sin\left(\frac{y}{\pi}\right)+3\right)^2 dy = 0.8 \times 259.53228\ldots$ $\int_0^h \pi\left(\sin\left(\frac{y}{\pi}\right)+3\right)^2 dy = 207.625824\ldots$ Solving using CAS $h = 4.9572\ldots$ cm i.e. Depth needs to be 4.96 cm for the glass to be 80% full.
Specific behaviours
✓ forms the correct expression to determine the volume when full ✓ determines the volume when full (or 80% full) ✓ forms the correct equation to solve for the 80% volume condition ✓ solves for the depth correct to 0.01 cm

Question 10 **(7 marks)**

(a) The sketch of the locus of a complex number z has been shown below. Write equations or inequalities in terms of z (without using $x = \mathrm{Re}(z)$ or $y = \mathrm{Im}(z)$) for the indicated locus. (4 marks)

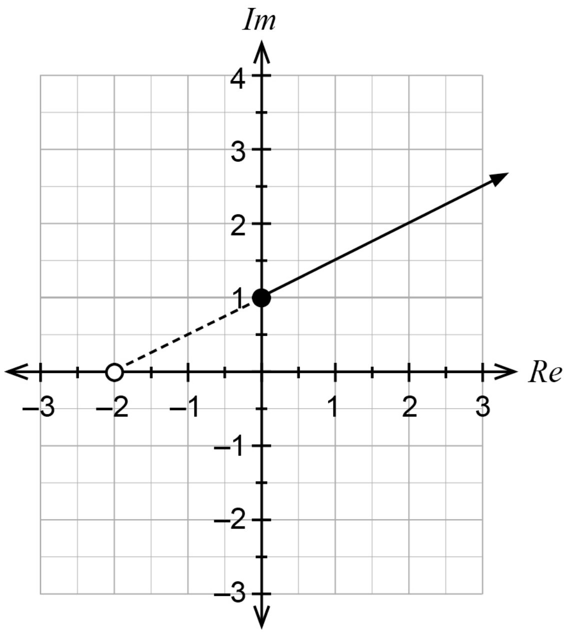


Solution
Arc is part of the circle $\lvert z-(1+2i)\rvert = 2$ such that $0 \le Arg\left(z-(1+2i)\right) \le \pi-\theta$ where $\tan\theta=\sqrt{3}$ i.e. $\theta=\frac{\pi}{3}$ $\therefore$ Locus is $\lvert z-(1+2i)\rvert = 2$, $0 \le Arg\left(z-(1+2i)\right) \le \frac{2\pi}{3}$
Specific behaviours
✓ writes the equation of the form $\lvert z-(1+2i)\rvert = r$ correctly ✓ states $r=2$ ✓ writes the inequality of the form $c \le Arg\left(z-(1+2i)\right) \le k$ ✓ uses correct trigonometry to determine the limits c, k

Question 10 (continued)

(b) Sketch the locus of the equation $|z+2| = |z-i| + \sqrt{5}$ in the Argand diagram below.

(3 marks)



Solution
Shown above. This equation can be interpreted as 'the distance from $z=-2$ is equal to $\sqrt{5}$ more than the distance from $z=i$'.
Specific behaviours
✓ indicates the locus as a ray (part of a line) ✓ indicates $(0,1)$ i.e. from $z=i$, is an element of the locus ✓ indicates the correct ray (correct slope from $z=-2$ to $z=i$)

Question 11 **(5 marks)**

Let z, w and u be complex numbers where:

$$w = (1+i)\overline{z} \qquad Arg(w) = \frac{\pi}{3} \qquad |w| = 2$$

$$u = \frac{z}{2-2i}$$

(a) Determine $Arg(u)$ exactly. (3 marks)

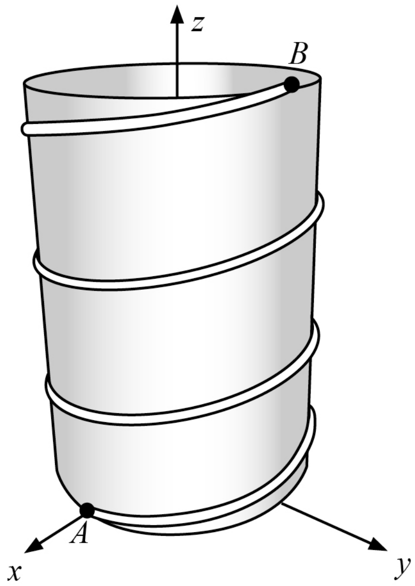
Solution
$Arg(w) = Arg(1+i) + Arg(\overline{z})$ $\frac{\pi}{3} = \frac{\pi}{4} - Arg(z)$ $\therefore \quad Arg(z) = -\frac{\pi}{12}$ $Arg(u) = Arg(z) - Arg(2-2i)$ $= -\frac{\pi}{12} - \left(-\frac{\pi}{4}\right)$ $= \frac{\pi}{6}$
Specific behaviours
✓ expresses relationships between arguments correctly ✓ determines $Arg(z)$ correctly ✓ determines $Arg(u)$ correctly

(b) Determine $|u|$ exactly. (2 marks)

Solution
$|w| = |1+i| \times |\overline{z}|$ $2 = \sqrt{2} \times |z|$ $\therefore \quad |z| = \sqrt{2}$ $|u| = \frac{|z|}{|2-2i|} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$
Specific behaviours
✓ expresses relationship between the modulus of numbers correctly ✓ determines $|u|$ correctly

Question 12 **(6 marks)**

A cylindrical shaped tower has a path that spirals upwards from the ground to an observation deck at point B as shown in the diagram below. The path begins at point A on the ground and finishes at point B at the top.



Let t = time in seconds that a tourist has been walking along the spiral path. The tourist takes 65π seconds to reach point B.

The tourist's position on this path at any time t is given by:

$$\underset{\sim}{r}(t) = \begin{pmatrix} 10\cos(0.1t) \\ 10\sin(0.1t) \\ 0.2t \end{pmatrix} \text{ metres.}$$

(a) Determine the height of the observation deck above the ground, correct to the nearest 0.01 metres. (1 mark)

Solution
Point B occurs when $t = 65\pi$.   Height $= z(B) = 0.2(65\pi) = 13\pi = 40.840...$ metres   Hence the height is 40.84 metres (to 0.01 metres).
Specific behaviours
✓ calculates the height correctly (no penalty for incorrect rounding)

(b) Determine the tourist's velocity $\underset{\sim}{v}(t)$. (2 marks)

Solution
$\underset{\sim}{v}(t) = \frac{d}{dt}\left(\underset{\sim}{r}(t)\right) = \begin{pmatrix} -\sin(0.1t) \\ \cos(0.1t) \\ 0.2 \end{pmatrix}$
Specific behaviours
✓ considers the derivative of the displacement vector   ✓ determines each component correctly

(c) Show that the tourist walks at a constant speed and determine this speed, correct to 0.01 metres per second. (3 marks)

Solution
$Speed(t) = \lvert \underset{\sim}{v}(t) \rvert = \sqrt{(-\sin(0.1t))^2 + (\cos(0.1t))^2 + (0.2)^2}$ $= \sqrt{\sin^2(0.1t) + \cos^2(0.1t) + 0.04}$ $= \sqrt{1 + 0.04}$ $= \sqrt{1.04} = 1.0198\ldots$ Hence the speed is constant and is 1.02 metres per second.
Specific behaviours
✓ forms the correct expression for the speed ✓ uses the trigonometric identity $\sin^2(x) + \cos^2(x) = 1$ ✓ evaluates the speed correctly to 0.01 metres per second

Question 13 **(4 marks)**

Solve the equation $z^4 = 8\sqrt{3}+8i$ giving exact solutions in the form $r cis\,\theta$ where $-\pi < \theta \le \pi$.

Solution
$z^4 = 8\sqrt{3}+8i \qquad r^2 = \left(8\sqrt{3}\right)^2 + 8^2$ $= 8^2(3)+8^2 = 4\left(8^2\right) \quad \therefore\ r = 2(8) = 16$ $\tan\theta = \dfrac{8}{8\sqrt{3}} \quad \therefore\ \theta = \dfrac{\pi}{6}$ Hence solve $z^4 = 16cis\left(\dfrac{\pi}{6}\right)$ Solutions are given by $z = 16^{\frac{1}{4}}cis\left(\dfrac{\frac{\pi}{6}+2\pi k}{4}\right)$ where $k = 0,1,2,3$.   Solutions are: $z_0 = 2cis\left(\dfrac{\pi}{24}\right)$ $z_1 = 2cis\left(\dfrac{13\pi}{24}\right)$ $z_2 = 2cis\left(\dfrac{13\pi}{24}+\dfrac{12\pi}{24}\right) = 2cis\left(-\dfrac{23\pi}{24}\right)$ $z_3 = 2cis\left(-\dfrac{11\pi}{24}\right)$
Specific behaviours
✓ determines the modulus correctly for $8\sqrt{3}+8i$ ✓ determines the argument correctly for $8\sqrt{3}+8i$ ✓ states one solution as $z = 2cis\left(\dfrac{\pi}{24}\right)$ ✓ states the correct arguments for the other 3 solutions (using $-\pi < \theta \le \pi$)

Question 14 **(5 marks)**

A particle travels in a straight line so that its velocity v cm per second and displacement x cm are related by the equation:

$$v = -0.2x$$

(a) Determine the acceleration a in terms of its displacement x. (2 marks)

Solution
Acceleration $a = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = (-0.2) \times (-0.2x)$ OR $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(0.02x^2\right) = 0.04x$ i.e. $a = 0.04x$
Specific behaviours
✓ uses the chain rule correctly to relate $\frac{dv}{dt}$ in terms of v ✓ obtains a correctly in terms of x

(b) Does the particle's motion constitute simple harmonic motion? Justify your answer. (1 mark)

Solution
Since $a = 0.04x$ then we could write $a = (0.2)^2 x$ But the condition for S.H.M. is that $a = -n^2 x$.   Hence the motion does NOT follow simple harmonic motion.
Specific behaviours
✓ justifies why the motion is NOT simple harmonic

It is known that the initial displacement of the particle is $x = 4$ cm.

(c) Determine, correct to the nearest 0.01 second, when the particle has a displacement of 2 cm. (2 marks)

Solution
We have $v = \frac{dx}{dt} = -0.2x$ $\therefore \int \frac{1}{x}\,dx = \int -0.2\,dt$ using separation of variables   i.e. $\ln\lvert x\rvert = -0.2t + c_1$ i.e. $x = e^{-0.2t+c} = k \times e^{-0.2t}$ Using $x(0) = 4$, then $4 = k\left(e^0\right)$ Hence $k = 4$ i.e. $x = e^{-0.2t+c} = 4e^{-0.2t}$ Solving $x(t) = 2$ yields $2 = 4e^{-0.2t}$ i.e. $t = 5\ln 2 = 3.47$ seconds (correct to 0.01 sec)
Specific behaviours
✓ determines the function $x(t)$ correctly   ✓ solves for t correct to 0.01 seconds

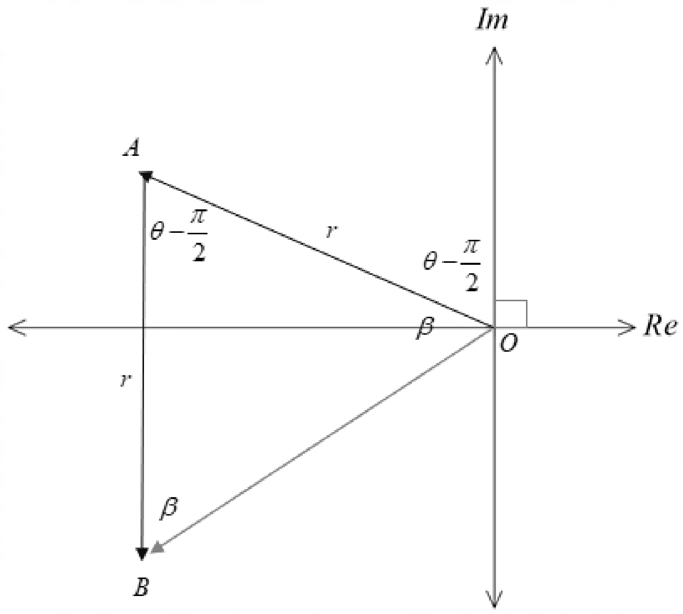
Question 15 **(6 marks)**

Let $z = r\,cis\,\theta$ be a complex number such that $\frac{\pi}{2} < \theta < \pi$.

(a) Express in terms of r and θ the complex number $\frac{\overline{z}}{-\sqrt{2}(i+1)}$. (3 marks)

Solution
$\frac{\overline{z}}{-\sqrt{2}(i+1)} = \frac{rcis(-\theta)}{2cis\left(-\frac{3\pi}{4}\right)} = \frac{r}{2}cis\left(\frac{3\pi}{4}-\theta\right)$
Specific behaviours
✓ writes the correct expression for $\overline{z}$ ✓ converts $-\sqrt{2}(i+1)$ into polar form correctly ✓ simplifies expression correctly in polar form

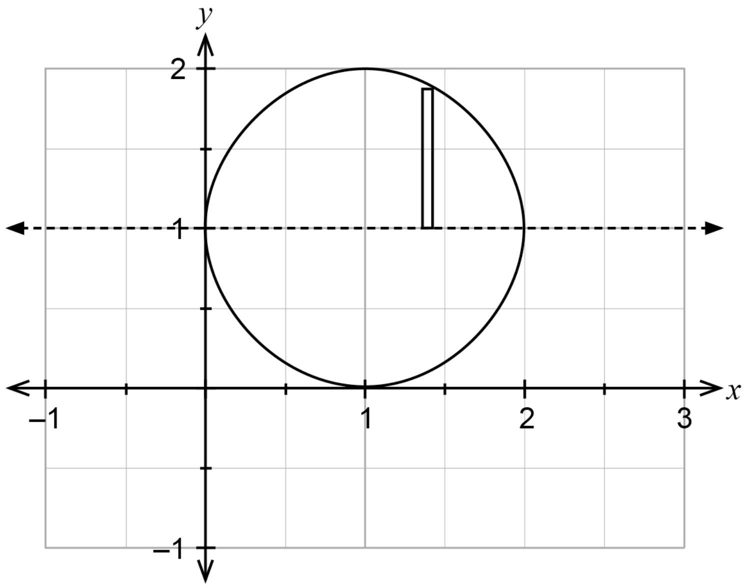
(b) Express $\alpha = Arg(z - ri)$ in terms of θ where $0 < \alpha < 2\pi$. (3 marks)



Solution
Vector $\overrightarrow{OB}$ represents $z - ri$. $s\angle OAB = \theta - \frac{\pi}{2}$ In isosceles ΔOAB, $OA = AB$ Hence $s\angle AOB = s\angle ABO = \beta$ We have $2\beta + \left(\theta - \frac{\pi}{2}\right) = \pi$ (Angle sum in ΔOAB) $\beta = \frac{3\pi}{4} - \frac{\theta}{2}$ Hence $\alpha = Arg(z - ri) = \theta + \beta = \frac{3\pi}{4} + \frac{\theta}{2}$
Specific behaviours
✓ draws the correct representation for $z - ri$ ✓ uses that $OA = AB$ to deduce congruent angles in ΔOAB ✓ deduces $\alpha = \frac{3\pi}{4} + \frac{\theta}{2}$

Question 16 **(5 marks)**

The curve shown below is given by the equation $(y-1)^2 = \sin\left(\frac{\pi x}{2}\right)$.



(a) Calculate the area, correct to 0.0001 square units, of the region that forms the interior of this curve. (3 marks)

Solution
The curve is symmetric about the line $y=1$.   Hence the height of the curve above $y=1$ is given by $y-1 = \sqrt{\sin\left(\frac{\pi x}{2}\right)}$ Hence using symmetry $Area = 2 \times \int_0^2 \sqrt{\sin\left(\frac{\pi x}{2}\right)}\,dx = 3.0510$ (4 d.p.)
Specific behaviours
✓ forms a definite integral using the correct limits   ✓ forms the integrand correctly   ✓ evaluates the integral correctly to 0.0001

(b) By using the answer to part (a), determine whether this curve is a circle. Explain your answer. (2 marks)

Solution
If the curve was a circle, with radius 1 unit then the $Area = \pi(1)^2 = 3.1416$ Hence since the Area from part (a) $\neq \pi$ then the curve is NOT a circle.
Specific behaviours
✓ compares answer to part (a) with the area of the circle radius 1 unit   ✓ concludes that the curve cannot be a circle

Question 17 **(8 marks)**

Members of a random sample of n shoppers at the El Cheepo shopping centre were asked by a consumer researcher how much they had spent in the shopping centre that day. Let μ denote the mean and σ the standard deviation of the amount spent. The standard deviation σ is known from previous research.

A 95% confidence interval for μ based on the sample is $150 \le \mu \le 200$ dollars.

(a) Determine the sample mean of this sample. (1 mark)

Solution
Sample mean $\overline{X} = \dfrac{150+200}{2} = 175$ Sample mean was \$175.
Specific behaviours
✓ calculates the sample mean correctly

(b) Based on this confidence interval, calculate the standard deviation of the sample mean, correct to 0.01. (3 marks)

Solution
Sample mean $\overline{X} = 175$ $\therefore\ 25 = 1.96 \times \sigma\left(\overline{X}\right)$ $\therefore\ \sigma\left(\overline{X}\right) = 12.7551\ldots$ i.e. Standard deviation of the sample mean was \$12.76 (2 d.p.)
Specific behaviours
✓ determines the critical z score for 95% confidence ✓ forms the equation relating the half-width of the interval and the standard deviation ✓ calculates the standard deviation of the sample mean correct to 0.01

The following week, the researcher again took a random sample of shoppers from the El Cheepo shopping centre, but this time the sample size was doubled.

(c) What is the probability that the difference between μ and the sample mean from this sample will be less than \$10? (4 marks)

Solution
We have that $\dfrac{\sigma}{\sqrt{n}} = 12.7551$ i.e. $\sigma = 12.7551\sqrt{n}$ For the larger sample $\sigma\left(\overline{X}\right) = \dfrac{12.7551\sqrt{n}}{\sqrt{2n}} = 9.0192\ldots$ Require $P\left(\left\|\overline{X}-\mu\right\|<10\right) = 2 \times P\left(0<z<\dfrac{10}{9.0192}\right) = 2\left(0.3665\ldots\right)$ $= 0.73$
Specific behaviours
✓ relates the standard deviations for the sample mean between the 2 samples ✓ determines the standard deviation of the sample mean for the sample size $2n$ ✓ forms the correct probability expression ✓ calculates the correct probability

Question 18 **(11 marks)**

The mass of chocolate that is placed into each biscuit produced by the BikkiesAreUs company has been observed to be normally distributed with mean $\mu =$ 7.5 grams and standard deviation $\sigma =$ 1.5 grams.

(a) Determine the probability, correct to 0.01, that the total amount of chocolate used for 50 biscuits is less than 365 grams. (4 marks)

Solution
Let $\overline{M}$ = the sample mean for the mass of chocolate per biscuit for 50 biscuits (g)
$= N\left(7.5,\ \sigma^2_{\overline{M}}\right)$ where $\sigma_{\overline{W}} = \dfrac{1.5}{\sqrt{50}} = 0.21213\ldots$
For a total of 365 g, the sample mean $\overline{M} = \dfrac{365}{50} = 7.3$ grams per biscuit
Require $P\left(\overline{M} < 7.3\right) = 0.1729\ldots = 0.17$
Specific behaviours
✓ states that the sample mean is a normal random variable ✓ states the correct parameters for the normal random variable ✓ calculates the sample mean correctly for the total 365 grams ✓ determines the correct probability (to 0.01)

(b) If the probability that the mean amount of chocolate used per biscuit differs from μ by less than 0.2 grams is 98%, determine n, the number of biscuits that need to be sampled. (3 marks)

Solution
$\sigma_{\overline{M}} = \dfrac{1.5}{\sqrt{n}}$ We require $P\left(-k < z < k\right) = 0.98$ this gives $k = 2.326$
Hence $2.326\left(\dfrac{1.5}{\sqrt{n}}\right) < 0.2$ Solving gives $n > 304.32$
i.e. we require at least 305 biscuits to have the sample mean differ by less than 0.2 grams
Specific behaviours
✓ uses the standard z score that represents 98% confidence ✓ forms the correct inequality/equation to solve for n ✓ states the correct minimum integer value for n

Question 18 (continued)

A competitor company called YouBeautChokkies produces similar biscuits. A sample of 144 biscuits was taken and it was found that the standard deviation of the mass of chocolate used in each biscuit was 1.8 grams and the total amount of chocolate used in the sample of 144 biscuits was 1.09 kg.

Charlie Chokka, a representative from the YouBeautChokkies company, stated that "we are using significantly more chocolate for each biscuit than BikkiesAreUs. If you want that real chocolate taste, then buy from us!"

(c) Perform the necessary calculations to comment on Charlie's claim. (4 marks)

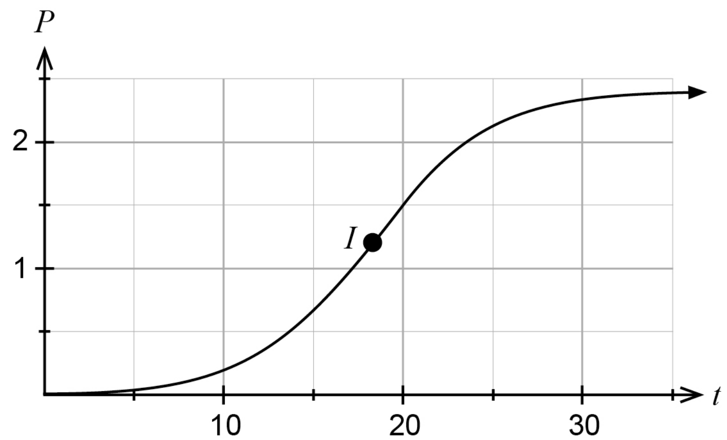
Solution
Let $\mu_Y =$ the population mean for the mass of chocolate per biscuit for the YouBeautChokkies company (grams) For the YouBeautChokkies total of 1090 grams, this gives $\overline{M} = 7.56944...$ grams The distribution for $\overline{M} \sim N\left(7.56944,\ \sigma_{\overline{M}}^2\right)$ where $\sigma_{\overline{M}} = \dfrac{1.8}{\sqrt{144}} = 0.15$ Confidence Interval for μ_Y 95% level : $7.56944 - 1.96\left(\sigma_{\overline{M}}\right) < \mu_Y < 7.56944 + 1.96\left(\sigma_{\overline{M}}\right)$ i.e. $7.2754 < \mu_Y < 7.8634$ Confidence Interval for μ_Y 99% level : $7.56944 - 2.576\left(\sigma_{\overline{M}}\right) < \mu_Y < 7.56944 + 2.576\left(\sigma_{\overline{M}}\right)$ $7.1830 < \mu_Y < 7.9558$ The BikkiesAreUs population mean $\mu =$ 7.5 is WITHIN the confidence interval using $\overline{M} = 7.56944$ and $\sigma = 1.8$. i.e. the claim is NOT vindicated. i.e. the YouBeautChokkies company are NOT using significantly more chocolate per biscuit than compared to BikkiesAreUs.
Specific behaviours
✓ determines the expected variation using $n = 144$ ✓ determines an appropriate confidence interval for the YouBeautChokkies population mean ✓ states that the BikkiesAreUs population mean 7.5 is within the confidence interval ✓ concludes correctly by writing a comment about the claim

Question 19 **(11 marks)**

The population $P(t)$ of sardines in an ocean, measured in million tonnes after t years, was modelled by the logistic equation:

$$P(t) = \frac{2.4}{1+239e^{-0.3t}}$$

The graph of this model is shown below. This graph contains a point of inflection at point I.



(a) Calculate the size of the sardine ocean population at $t = 0$. (2 marks)

Solution
$P(0) = \dfrac{2.4}{1+239} = 0.01$ million tonnes3 i.e. The population number is 10 000 tonnes at $t = 0$.
Specific behaviours
✓ substitutes $t = 0$ into the logistic equation and calculates correctly ✓ states the answer in tonnes OR states the units of the answer (million tonnes)

(b) Rewrite the logistic equation in the form $\dfrac{dP}{dt} = rP(k-P)$, stating clearly the values for r and k. (2 marks)

Solution
$k =$ limiting population value $\lim\limits_{t\to\infty} P(t) = 2.4$ Hence $k = 2.4$ From defining rule for $P(t)$, $rk = 0.3$ Hence $r = \dfrac{0.3}{2.4} = 0.125$ i.e. $\dfrac{dP}{dt} = 0.125P(2.4-P)$
Specific behaviours
✓ determines value of k correctly ✓ determines value of r correctly

Question 19 (continued)

(c) When the sardine population is 500 000 tonnes, use the technique of increments to calculate the approximate change in population in the next month. (3 marks)

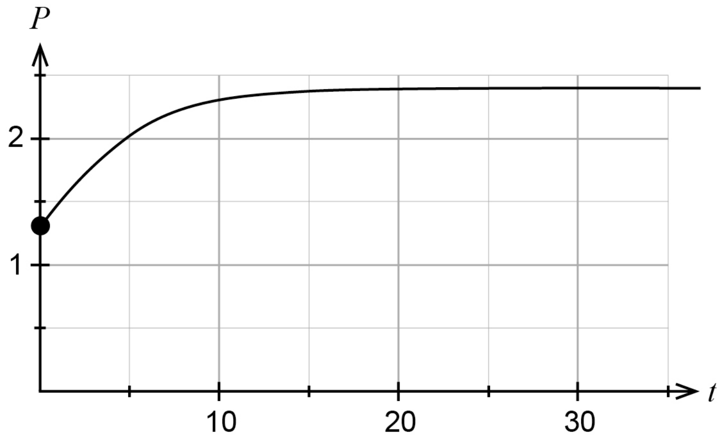
Solution
When $P = 0.5$ million tonnes and $\Delta t = \dfrac{1}{12}$ years We have $\Delta P \approx \left(\dfrac{dP}{dt}\right) \times \Delta t = 0.125(0.5)(2.4-0.5) \times \dfrac{1}{12}$ $= (0.11875) \times \dfrac{1}{12} = 0.0098958...$ million tonnes i.e. Sardine population grows by approximately 9896 tonnes (9900 tonnes)
Specific behaviours
✓ states the correct values for P and Δt ✓ forms the correct expression for ΔP ✓ calculates the correct value for ΔP (units are not necessary)

(d) Determine the maximum rate of growth of the sardine population. (2 marks)

Solution
Maximum occurs at the value of $P = \dfrac{k}{2} = 1.2$ million tonnes (point of inflection) Note: This occurs when $t = 18.254...$ years. $\therefore \dfrac{dP}{dt} = 0.125(1.2)(2.4-1.2) = 0.18$ million tonnes/year Hence the maximum rate of growth is 180 000 tonnes per year.
Specific behaviours
✓ determines the value of P that yields the maximum rate of growth ✓ calculates the rate of growth correctly stating correct units

Suppose that the initial population of sardines was 1.3 million tonnes.

(e) Assuming that the rate of growth is still given by $\frac{dP}{dt} = rP(k-P)$ sketch the graph of the population growth on the axes below. Explain your graph. (2 marks)

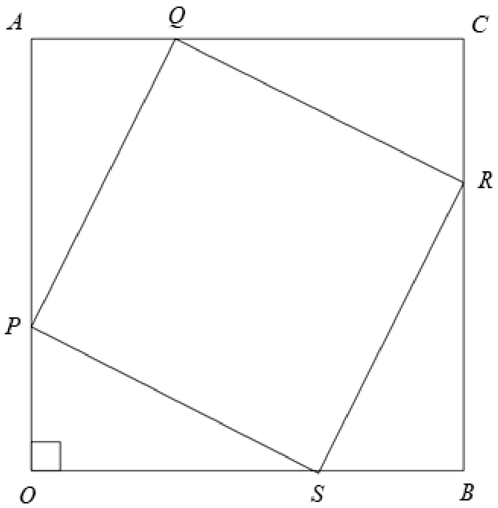


Solution
Sketch shown above. Since $P(0) > \frac{k}{2} = 1.2$ then $\frac{dP}{dt}$ will be decreasing. $\therefore$ Graph will NOT exhibit any point of inflection and will be concave DOWN as it approaches the horizontal asymptote $P = 2.4$.
Specific behaviours
✓ indicates a curve that is always concave down approaching $P = 2.4$ ✓ justifies using the initial population value $P(0) > 1.2$

Question 20 **(9 marks)**

Consider square $OACB$ where point O is the origin. Let the position vectors for points A, B be defined as $\underset{\sim}{a}, \underset{\sim}{b}$ respectively i.e. $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.

Let points P, Q, R and S be defined so that $\overrightarrow{OP} = k\,\underset{\sim}{a}$, $\overrightarrow{AQ} = k\,\underset{\sim}{b}$, $\overrightarrow{RC} = k\,\underset{\sim}{a}$ and $\overrightarrow{SB} = k\,\underset{\sim}{b}$ where $0 \le k \le 1$. This means that points P, Q, R and S are positioned along their respective sides in equal proportion.



(a) Using vector methods, prove that the size of $\angle PQR = 90^\circ$. (5 marks)

Solution
Vectors $\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AQ} = (1-k)\underset{\sim}{a} + k\underset{\sim}{b}$ $\overrightarrow{QR} = \overrightarrow{QC} + \overrightarrow{CR} = (1-k)\underset{\sim}{b} + (-k\underset{\sim}{a}) = (1-k)\underset{\sim}{b} - k\underset{\sim}{a}$
Consider $\overrightarrow{PQ} \bullet \overrightarrow{QR} = \left((1-k)\underset{\sim}{a} + k\underset{\sim}{b}\right) \bullet \left((1-k)\underset{\sim}{b} - k\underset{\sim}{a}\right)$ $= (1-k)^2\,\underset{\sim}{a} \bullet \underset{\sim}{b} - k(1-k)\underset{\sim}{a} \bullet \underset{\sim}{a} + k(1-k)\underset{\sim}{b} \bullet \underset{\sim}{b} - k^2\underset{\sim}{b} \bullet \underset{\sim}{a}$
Since $s\angle AOB = 90^\circ$ then $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$
Then $\overrightarrow{PQ} \bullet \overrightarrow{QR} = -k(1-k)\underset{\sim}{a} \bullet \underset{\sim}{a} + k(1-k)\underset{\sim}{b} \bullet \underset{\sim}{b} = -k(1-k)\lvert\underset{\sim}{a}\rvert^2 + k(1-k)\lvert\underset{\sim}{b}\rvert^2$
But since side lengths are equal then $\lvert\underset{\sim}{a}\rvert = \lvert\underset{\sim}{b}\rvert$
Then $\overrightarrow{PQ} \bullet \overrightarrow{QR} = -k(1-k)\lvert\underset{\sim}{a}\rvert^2 + k(1-k)\lvert\underset{\sim}{a}\rvert^2 = 0$
Hence the size of $\angle PQR = 90^\circ$ as required.
Specific behaviours
✓ forms correct expressions for vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$
✓ forms the dot product of vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ using correct notation
✓ expands the dot product expression correctly to obtain 4 terms
✓ uses the dot product $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$ since $OACB$ is given as a square
✓ uses the information $\lvert\underset{\sim}{a}\rvert = \lvert\underset{\sim}{b}\rvert$ to obtain the dot product ZERO

Now suppose that in square $OACB$, it is known that $OA = 10$ cm and that point P is moving away from the origin at a speed of 0.2 cm per second. This means that points Q, R and S are moving at the same speeds along their respective sides.

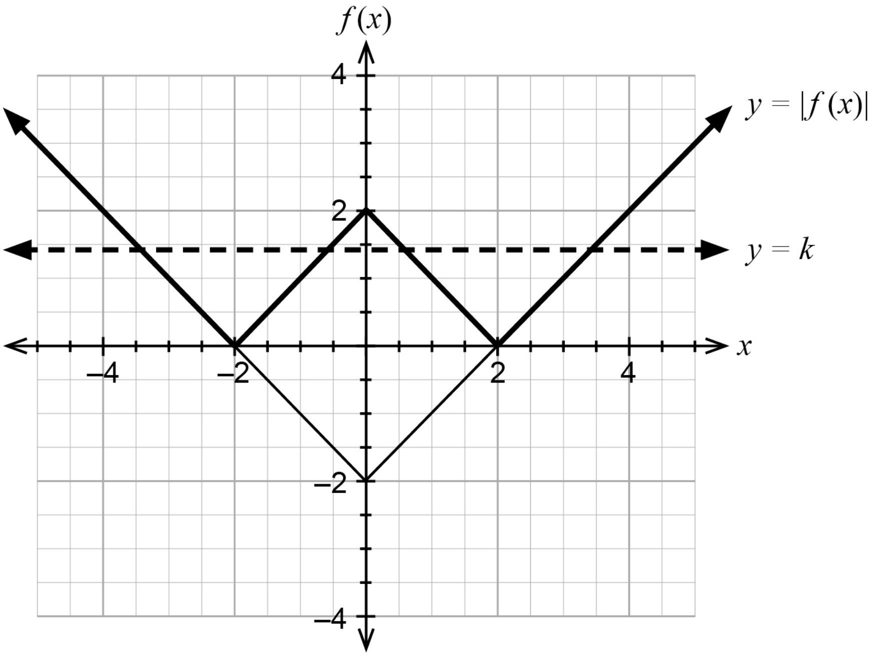
Let $x =$ the distance OP.

(b) Determine the rate at which the area of square $PQRS$ is changing when $x = 3$ cm. (4 marks)

Solution
Area $\Delta OPS \;=\; \frac{1}{2}(x)(10-x)$ Hence area $PQRS \;\; A \;=\; (10)(10) \;-\; 4\times\frac{1}{2}(x)(10-x)$ $\qquad = \; 100-20x+2x^2$ $\frac{dA}{dt} \;=\; \frac{dA}{dx} \times \frac{dx}{dt} \;=\; (-20+4x)\frac{dx}{dt} \;=\; (-20+4(3))(0.2)$ $\qquad = \; -1.6$ cm^2/sec Hence the area is decreasing at a rate of 1.6 cm^2/sec.
Specific behaviours
✓ forms the correct expression for the area of $PQRS$ in terms of x ✓ applies the chain rule to obtain $\frac{dA}{dt}$ in terms of x and $\frac{dx}{dt}$ correctly ✓ substitutes $x = 3$ and $\frac{dx}{dt} = 0.2$ correctly ✓ calculates correctly and states the correct units for the rate of change of area

Question 21 **(5 marks)**

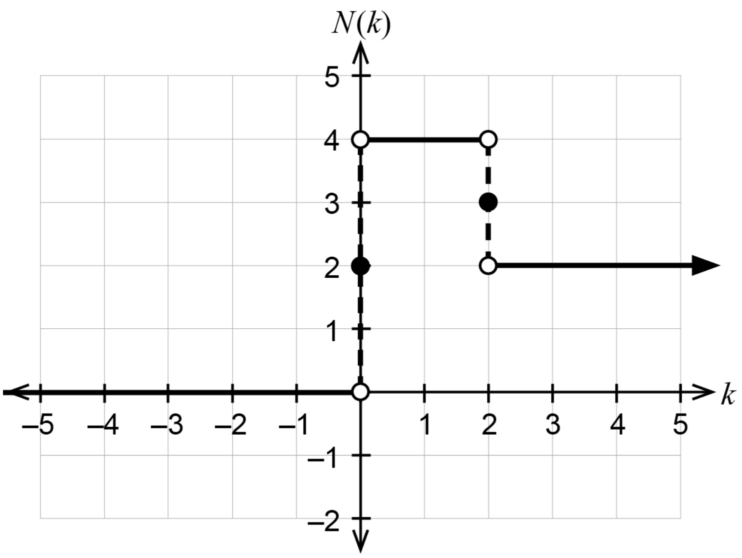
The sketch of the graph of $y = f(x)$ is shown below.



Consider the equation $|f(x)| = k$ where k is any real constant.

Define function $N(k)$ = the number of real solutions to the equation $|f(x)| = k$.

Sketch the graph of function $N(k)$ on the axes below.



Solution
Shown above as a piecewise defined function.
Specific behaviours
✓ indicates the graph of $y = \lvert f(x) \rvert$ is intersected with the horizontal line $y = k$ ✓ indicates $(0,2)$ and $(2,3)$ ✓ indicates $N(k) = 0$ for $k < 0$ ✓ indicates $N(k) = 4$ for $0 < k < 2$ ✓ indicates $N(k) = 2$ for $k > 2$

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