Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS METHODS

Calculator-assumed

ATAR course examination 2017

Marking Key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2017/69886

Section Two: Calculator-assumed **65% (99 Marks)**

Question 10 **(3 marks)**

Use the quotient rule to show that $\frac{d}{dx}\tan(x) = \frac{1}{\cos^2(x)}$.

Solution
$\frac{d}{dx}\tan(x) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)$ $= \frac{\cos(x)\times\cos(x) - \sin(x)\times(-\sin(x))}{\cos^2(x)}$ $= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}$ $= \frac{1}{\cos^2(x)} \quad \left\{\text{since } \cos^2(x)+\sin^2(x) \equiv 1\right\}$
Specific behaviours
✓ writes tangent as a ratio of sine and cosine   ✓ demonstrates use of the quotient rule   ✓ states and uses the Pythagorean identity to simplify result

Question 11 **(9 marks)**

A pizza shop estimates that the time X hours to deliver a pizza from when it is ordered is a continuous random variable with probability density function given by

$$f(x)=\begin{cases}\dfrac{4}{3}-\dfrac{2}{3}x\ , & 0<x<1\\ 0\ , & \text{otherwise.}\end{cases}$$

(a) What is the probability of a pizza being delivered within half an hour of being ordered? (2 marks)

Solution
$P(X<0.5)=\int_0^{0.5}\left(\frac{4}{3}-\frac{2}{3}x\right)dx=\frac{4}{3}x-\frac{1}{3}x^2\Big\|_0^{0.5}=\frac{2}{3}-\frac{1}{12}=\frac{7}{12}\approx 0.5833$ OR $P(X<0.5)=\text{Area of trapezium}=\frac{1}{4}\left(\frac{4}{3}+1\right)=\frac{7}{12}\approx 0.5833$
Specific behaviours
✓ writes correct integral (or area) expression for probability   ✓ calculates probability correctly

(b) Calculate the mean delivery time to the nearest minute. (3 marks)

Solution
$E(X)=\int_0^{1}x\left(\frac{4}{3}-\frac{2}{3}x\right)dx=\frac{2}{3}x^2-\frac{2}{9}x^3\Big\|_0^{1}=\frac{2}{3}-\frac{2}{9}=\frac{4}{9}\approx 0.4444$ That is, 27 minutes.
Specific behaviours
✓ writes the correct integral for the mean   ✓ calculates the mean correctly   ✓ converts to minutes

Question 11 (continued)

(c) Calculate the standard deviation of the delivery time to the nearest minute. (4 marks)

Solution
$$\mathrm{Var}(X)=\int_0^1\left(\frac{4}{3}-\frac{2}{3}x\right)\left(x-\frac{4}{9}\right)^2dx$$ $$=0.0802$$ OR $$E\left(X^2\right)=\int_0^1 x^2\left(\frac{4}{3}-\frac{2}{3}x\right)dx=\frac{4}{9}x^3-\frac{1}{6}x^4\Big\|_0^1=\frac{4}{9}-\frac{1}{6}=\frac{10}{36}=\frac{5}{18}\approx 0.2778$$ So $\mathrm{Var}\left(X\right)=\frac{5}{18}-\frac{16}{81}=\frac{13}{162}\approx 0.0802$ $$\sigma=\sqrt{0.0802}\approx 0.2833$$ That is, 17 minutes.
Specific behaviours
✓ calculates $E(X^2)$ correctly or states integral for VAR ✓ calculates the variance correctly ✓ calculates standard deviation correctly ✓ converts to minutes

Question 12 **(13 marks)**

The Slate Tablet Company produces a variety of electronic tablets. It wants to gather information on consumers' interest in its tablets.

(a) In each of the following cases, comment, giving reasons, whether or not the proposed sampling method introduces bias.

(i) A Slate Tablet Company representative stood outside an electronics store on a Saturday morning and asked people entering the store 'If you were to purchase an electronic tablet would you choose a Slate Tablet or an inferior brand?' (2 marks)

Solution
The method is biased due to: • the people being asked a leading question • the specific time and location used for the survey.
Specific behaviours
✓ states method biased with reason ✓ states a correct reason

(ii) Fifteen hundred randomly selected mobile phone numbers were telephoned and people were asked 'Which brand of electronic tablet do you prefer?' (2 marks)

Solution
In this case the question is not biased, however, only mobile phone users were selected causing bias. Also many of these people may just hang up.
Specific behaviours
✓ states the method is biased with reason ✓ states a correct reason

A common problem with a particular tablet is screen failure. The manufacturer of Slate Tablets has found that 1% of their its screens will fail within three years. A sample of 200 tablets is taken. Let the random variable X denote the number of tablets that have screen failure within three years in the sample of 200.

(b) What is the distribution of X ? (2 marks)

Solution
$X \sim Bin(200, 0.01)$
Specific behaviours
✓ identifies the binomial distribution ✓ specifies correct parameters

(c) What is the probability that more than four tablets will have screen failure within three years? (2 marks)

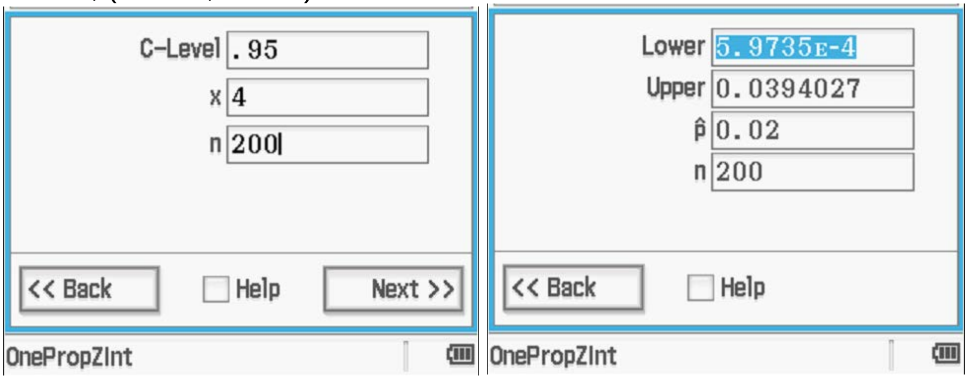
Solution
$P(X > 4) = 1 - P(X \leq 4) = 1 - 0.9482 = 0.0517$
Specific behaviours
✓ uses correct parameters for binomial ✓ calculates correct probability

Question 12 (continued)

In a random sample of 200 Slate Tablets, four of them had screen failure within three years.

(d) Calculate an approximate 95% confidence interval for the proportion of tablets that have screen failure within three years. Give your answer to four decimal places. (3 marks)

Solution
$\hat{p} = \frac{4}{200} = 0.02$ 95% confidence interval $= \left(0.02 - 1.96 \times \sqrt{\frac{0.02\times0.98}{200}}, 0.02 + 1.96 \times \sqrt{\frac{0.02\times0.98}{200}}\right)$ That is, (0.0006,0.0394).



Specific behaviours
✓ calculates correct sample proportion   ✓ calculates standard error correctly   ✓ calculates interval correctly

(e) The company's quality control department wants the proportion of tablets with faulty screens to be between 0.5% and 1%. Based on your confidence interval, decide whether the quality control department is meeting its target. Justify your decision. (2 marks)

Solution
The lower end of the confidence interval is below 0.005, so the lower target is met. However, the higher end is above 0.01, so the upper target is not met.

Specific behaviours
✓ refers to targets with reference to confidence interval   ✓ states decision

Question 13 **(9 marks)**

Ravi runs a dice game in which a player throws two standard six-sided dice and the sum of the uppermost faces is calculated. If the sum is less than five, the player wins $20. If the sum is greater than eight, the player wins $10. Otherwise the player receives no money.

(a) Complete the table below. (2 marks)

Solution

Amount won	20	10	0
Probability	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{20}{36}$

Specific behaviours

✓ completes top row correctly
✓ determines probabilities correctly

(b) What is the expected amount of money won by a player each time they play? (2 marks)

Solution

Let the random variable X be the amount of money won by a player:

$$E(X) = 20\times\frac{6}{36}+10\times\frac{10}{36}$$
$$=\frac{220}{36}$$
$$=\$6.11$$

Specific behaviours

✓ writes a calculation for expected value
✓ determines expected value

(c) Liu Yang decides to play the game. If Ravi charges her $5 to roll two dice, who is likely to be better off in the long-term? Explain. (3 marks)

Solution

$$Expected\ \ payout = 6.11-5$$
$$=1.11$$

Lui Yang is better off in the long term.
In the long term Liu Yang will likely win $1.11 per game.

Specific behaviours

✓ determines new expected payout
✓ states Lui Yang better off
✓ explains the meaning of the expected payout

Question 13 (continued)

(d) If Ravi wants to make a long-term profit per game of 20% of what he charges, what should he charge a player to roll the two dice? (2 marks)

Solution
Let amount to be paid be $\$P$ $E(X) = -0.2P$ $-0.2P = 20\times\frac{6}{36} + 10\times\frac{10}{36} - P$ $0.8P = 6.11$ $P = \$7.64$
Specific behaviours
✓ equates $E(X)$ to $-0.2P$ ✓ solves to give P

Question 14 **(9 marks)**

Let $f(x) = x\ln(x+3)$.

(a) Use calculus to locate and classify all the stationary points of $f(x)$ and find any points of inflection. (5 marks)

Solution

$$\frac{df}{dx} = \frac{x}{x+3} + \ln(x+3)$$

For SPs: $0 = \dfrac{x}{x+3} + \ln(x+3)$

$$x = -1.1454$$

$$y = -0.7075$$

$$\frac{d^2 f}{dx^2} = \frac{x+3-x}{(x+3)^2} + \frac{1}{x+3} = \frac{x+6}{(x+3)^2}$$

$$\left.\frac{d^2 f}{dx^2}\right|_{x=-1.145449} = 1.411 = \text{positive}$$

therefore minimum

For POI: $\dfrac{d^2 f}{dx^2} = 0$

$\therefore$ POI when $x = -6$

So no POI as the function is undefined for $x \le -3$.

$$\text{solve}\left(\ln(x+3) + \frac{x}{x+3} = 0, x\right)$$

$$\{x = -1.145449281\}$$

$$\frac{d}{dx}\left(\ln(x+3) + \frac{x}{x+3}\right)$$

$$\frac{x+6}{(x+3)^2}$$

$$\frac{d^2}{dx^2}(f(x))\,|x=-1.145449281$$

$$1.411469872$$

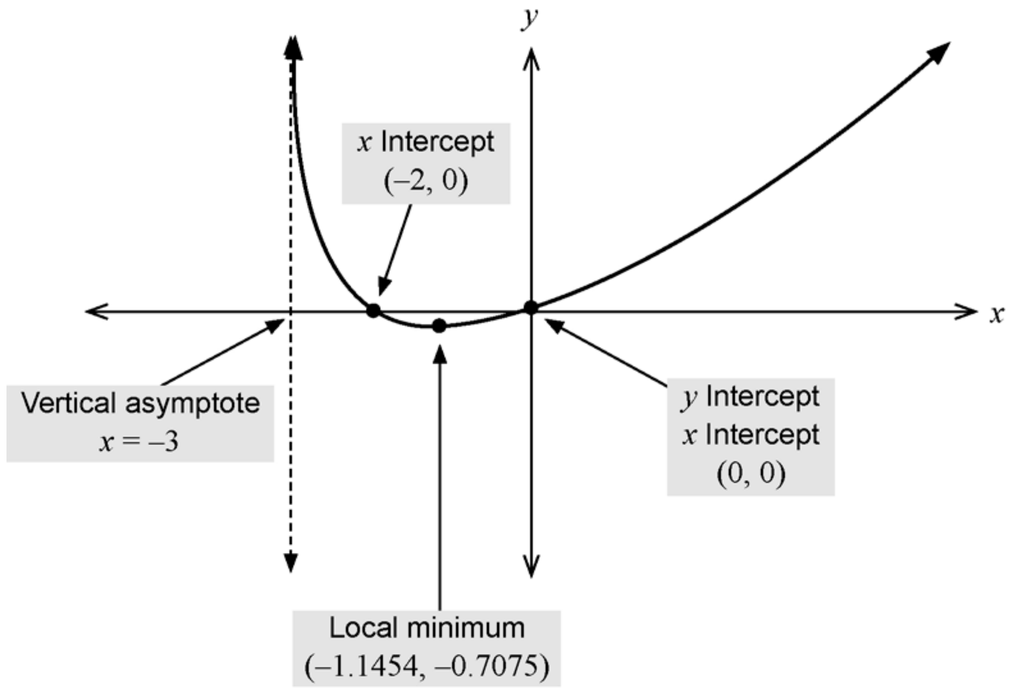
Specific behaviours

- ✓ differentiates correctly
- ✓ finds the critical point
- ✓ finds y co-ordinate and justifies minimum
- ✓ finds second derivative
- ✓ rejects point of inflection

Question 14 (continued)

(b) On the axes provided sketch the graph of $f(x)$, labelling all key features. (4 marks)

Solution



Specific behaviours

✓ shows vertical asymptote
✓ shows minimum
✓ shows intercepts
✓ has correct shape of graph

Question 15 **(10 marks)**

The volume $V(h)$ in cubic metres of liquid in a large vessel depends on the height h (metres) of the liquid in the vessel and is given by

$$V(h)=\int_0^h e^{\left(-\frac{x^2}{100}\right)}dx,\ \ 0\le h\le 15.$$

(a) Determine $\dfrac{dV}{dh}$ when the height is 0.5 m. (2 marks)

Solution
$V'(h)=e^{\left(-\frac{h^2}{100}\right)}$ So $V'(0.5)=\ e^{-0.0025}=0.9975m^3/m$
Specific behaviours
✓ uses FTC   ✓ obtains correct value for the rate of change

(b) What is the meaning of your answer to Part (a)? (1 mark)

Solution
It means the rate of change of the volume with respect to height when the height has reached 0.5 metres.
Specific behaviours
✓ states meaning

(c) The height h of the liquid depends on time t (seconds) as follows:

$$h(t)=3t^2-t+4, t\ge 0.$$

(i) Determine $\dfrac{dh}{dt}$ when the height is 6 m. (2 marks)

Solution
Now $h(t)=3t^2-t+4=6\Longrightarrow 3t^2-t-2=0\Longrightarrow(3t+2)(t-1)=0$ So $t=1\ s$. Then $\dfrac{dh}{dt}=6t-1$ $\left.\dfrac{dh}{dt}\right\rvert_{t=1}=6(1)-1=5$ m/s
Specific behaviours
✓ differentiates h wrt t correctly   ✓ state equation for time and substitutes values correctly

(ii) Use the chain rule to determine $\dfrac{dV}{dt}$ when the height is 6 m. (2 marks)

Solution
$\dfrac{dV}{dt}=\dfrac{dV}{dh}\times\dfrac{dh}{dt}$ $=e^{-\frac{6^2}{100}}\times 5$ ≈ 3.488 m^3/s
Specific behaviours
✓ demonstrates use of the chain rule ✓ substitutes values correctly to determine rate of change

(iii) Given the volume of the liquid at 2 seconds is 8.439 m^3, use the incremental formula to estimate the volume 0.1 second later. (3 marks)

Solution
$h(2)=3(2)^2-2+4=14$ $\dfrac{\delta V}{\delta t}\approx\dfrac{dV}{dt}$ $\delta V\approx e^{-\frac{14^2}{100}}\times 11\times\delta t$ $\approx 1.54944\times 0.1$ ≈ 0.155 $V\left(t=2.1\right)\approx 8.439+0.155$ ≈ 8.594 m^3
Specific behaviours
✓ determines $h(2)$ ✓ uses incremental formula to approx. dV ✓ estimates new V

Question 16 **(8 marks)**

A group of biologists has decided that colonies of a native Australian animal are in danger if their populations are less than 1000. One such colony had a population of 2300 at the start of 2011. The population was growing continuously such that $P = P_0 e^{0.065t}$ where P is the number of animals in the colony t years after the start of 2011.

(a) Determine, to the nearest 10 animals, the population of the colony at the start of 2014. (2 marks)

Solution
$P(t) = 2300e^{0.065t}$ $P(3) = 2300e^{0.065(3)}$ $= 2795.2$ ≈ 2800
Specific behaviours
✓ determines equation for P ✓ determines population correct to nearest 10

(b) Determine the rate of change of the colony's population when $t = 2.5$ years. (2 marks)

Solution
$\frac{dP}{dt} = 0.065 \times 2300e^{0.065t}$ $\left.\frac{dP}{dt}\right\|_{t=2.5} = 175.879$ ≈ 176 animals/year
Specific behaviours
✓ determines derivative ✓ determines rate at 2.5 years

(c) At the beginning of 2017, a disease caused the colony's population to decrease continuously at the rate of 8.25% of the population per year. If this rate continues, when will the colony become 'in danger'? Give your answer to the nearest month. (4 marks)

Solution
$P(6) = 2300e^{0.065(6)}$ ≈ 3397 Population from 2017: $P(t) = 3397e^{-0.0825t}$ $1000 = 3397e^{-0.0825t}$ $t = 14.8$ October 2031
Specific behaviours
✓ determines population at the beginning of 2017 ✓ states new population equation ✓ solves for t ✓ determines correct month and year

Question 17 **(6 marks)**

A beverage company has decided to release a new product. 'Joosilicious' is to be sold in 375 mL cans that are perfectly cylindrical. {Hint: $1mL = 1cm^3$}

(a) If the cans have a base radius of x cm show that the surface area of the can, S, is given by: $S = 2\pi x^2 + \dfrac{750}{x}$. (2 marks)

Solution
$375 = \pi x^2 h$ $\therefore h = \dfrac{375}{\pi x^2}$ $S = 2\pi x^2 + 2\pi x h$ $= 2\pi x^2 + 2\pi x\left(\dfrac{375}{\pi x^2}\right)$ $= 2\pi x^2 + \dfrac{750}{x}$
Specific behaviours
✓ uses volume formula to determine h in terms of x ✓ demonstrates substitution of h into surface area formula and simplifies to show result

(b) Using calculus methods, and showing full reasoning and justification, find the dimensions of the can that will minimise its surface area. (4 marks)

Solution
$S = 2\pi x^2 + \dfrac{750}{x}$ $\dfrac{dS}{dx} = 4\pi x - \dfrac{750}{x^2}$ $0 = 4\pi x - \dfrac{750}{x^2}$ $x = 3.908 \text{ cm}$ $\dfrac{d^2S}{dx^2} = 4\pi + \dfrac{1500}{x^3}$ $\dfrac{d^2S}{dx^2}\Big\vert_{x=3.908} = +ve\ (37.7) \Rightarrow \text{Min}$ When $x = 3.908$, $h = 7.816$ Cans have a radius of 3.9 cm and a height of 7.8 cm to minimise surface area
Specific behaviours
✓ determines first derivate ✓ equates to zero to find x ✓ justifies minimum with second derivative or other suitable method ✓ states dimensions of can

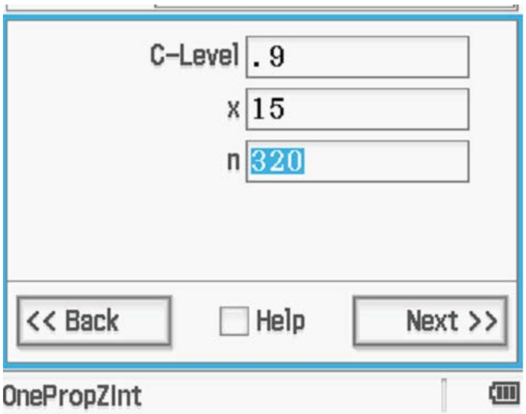
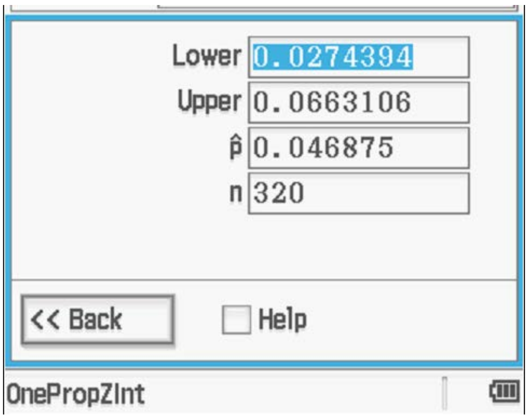
Question 18 **(11 marks)**

Alex is a beekeeper and has noticed that some of the bees are very sleepy. She takes a random sample of 320 bees and finds that 15 of them are indeed so-called *lullabees* that fall asleep easily.

(a) Calculate the sample proportion of lullabees. (1 mark)

Solution
$\dfrac{15}{320} = 0.046875$
Specific behaviours
✓ calculates proportion

(b) Determine a 90% confidence interval for the true proportion of lullabees, rounded to four decimal places. (3 marks)

Solution
$\left[\dfrac{15}{320} - 1.645\sqrt{\dfrac{\left(\frac{15}{320}\right)\left(1-\frac{15}{320}\right)}{320}}, \dfrac{15}{320} + 1.645\sqrt{\dfrac{\left(\frac{15}{320}\right)\left(1-\frac{15}{320}\right)}{320}}\right] \approx [0.0274, 0.0663]$
 
Specific behaviours
✓ uses $Z \approx 1.645$, ✓ calculates confidence interval ✓ rounds to four decimal places

(c) What is the margin of error in the above estimate? (2 marks)

Solution
$1.645\sqrt{\dfrac{\left(\frac{15}{320}\right)\left(1-\frac{15}{320}\right)}{320}} = 0.0194$
Specific behaviours
✓ substitutes into formula ✓ calculates standard error

Question 18 (continued)

It turns out that the true proportion of lullabees is 0.02.

(d) Now that Alex knows this, she decides to take a new sample.

(i) Suppose a new sample of 290 bees was taken. Given that the true proportion of lullabees is 0.02, what is the probability that the sample proportion in this new sample is at most 0.03? (3 marks)

Solution
normCDf$\left(-10, 0.03, \sqrt{\dfrac{0.02*0.98}{290}}, 0.02\right.$ 0.8880808029   i.e. a probability of approximately 0.89.
Specific behaviours
✓ CDF up to 0.03   ✓ determines standard deviation   ✓ calculates probability

(ii) If Alex takes a larger sample, will the above probability increase or decrease? Explain. (2 marks)

Solution
Increase. The larger sample size will result in a smaller standard deviation. With a less dispersed distribution the required probability will increase.
Specific behaviours
✓ states increase and SD decreased   ✓ states lower SD will give less dispersion and therefore higher probability

Question 19 **(12 marks)**

A global financial institution transfers a large aggregate data file every evening from offices around the world to its Hong Kong head office. Once the file is received it must be processed in the company's data warehouse. The time T required to process a file is normally distributed with a mean of 90 minutes and a standard deviation of 15 minutes.

(a) An evening is selected at random. What is the probability that it takes more than two hours to process the file? (2 marks)

Solution
$T \sim N(90, 15^2)$ so $P(T > 120) = P\left(Z > \frac{120-90}{15}\right) = P(Z > 2) = 0.0228$
Specific behaviours
✓ writes correct probability statement ✓ calculates correct probability

(b) What is the probability that the process takes more than two hours on two out of five days in a week? (3 marks)

Solution
Let the random variable X denote the number of days out of 5 that the process takes more than 2 hours. Then $X \sim Bin(5,0.0228)$. $P(X = 2) = \binom{5}{2} 0.0228^2(1 - 0.0228)^3 = 0.00485$
Specific behaviours
✓ identifies binomial distribution ✓ uses correct parameters for binomial ✓ calculates correct probability

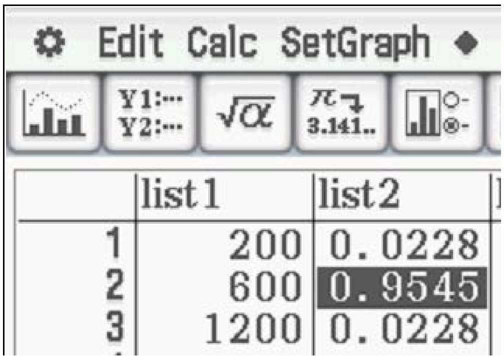
Question 19 (continued)

The company is considering outsourcing the processing of the files.

(c) (i) A quotation for this job from an IT company is given in the table below. Complete the table. (1 mark)

Solution

Job Duration T (minutes)	$T \leq 60$	$60 < T < 120$	$T \geq 120$
Probability	**0.0228**	**0.9545**	**0.0228**
Cost Y (\$)	200	600	1200



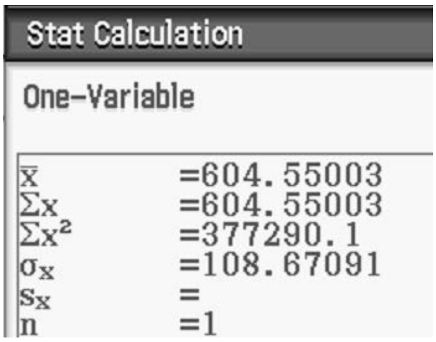
Specific behaviours

✓ calculates the probabilities correctly

(ii) What is the mean cost? (2 marks)

Solution

The probability distribution of Y is given below.



mean cost = $200 \times 0.0228 + 600 \times 0.9545 + 1200 \times 0.0228$

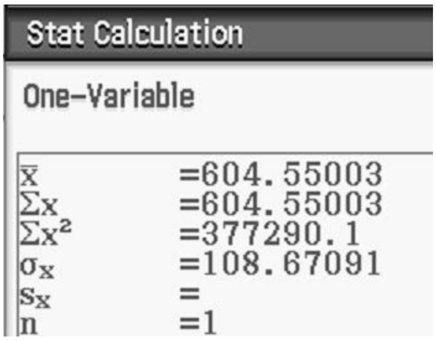
$= \$604.55$

Specific behaviours

✓ writes an expression for the mean cost per file

✓ calculates the mean correctly

(iii) Calculate the standard deviation of the cost. (2 marks)

Solution

$E(Y^2) = 200^2 \times 0.0228 + 600^2 \times 0.9545$ $+ 1200^2 \times 0.0228 = 377\,290$ So $\mathrm{Var}(Y) = 377\,290 - 604.55^2 =$ $11\,809.36668$ and $\sigma_X = \$108.67$
Specific behaviours
✓ calculates $E(X^2)$ correctly ✓ calculates the standard deviation correctly

(iv) In the following year, the cost (currently $\$Y$) will increase due to inflation and also the introduction of an additional fixed cost, so the new cost $\$N$ is given by: $N = aY + b$. In terms of a and/or b, state the mean cost in the following year and the standard deviation of the cost in the following year. (2 marks)

Solution
New Mean $= 604.55a + b$ New SD $= 108.67a$
Specific behaviours
✓ states new mean correctly ✓ states new SD correctly

Question 20 **(9 marks)**

A model train travels on a straight track such that its acceleration after t seconds is given by $a(t) = pt - 13$ cm/s^2, $0 \le t \le 10$, where p is a constant.

(a) Determine the initial acceleration of the model train. (1 mark)

Solution
$a(0) = -13 \quad \text{cm/s}^2$
Specific behaviours
✓ determines initial acceleration

The model train has an initial velocity of 5 cm/s. After 2 seconds it has a displacement of −50 cm. A further 4 seconds later its displacement is 178 cm.

(b) Determine the value of the constant p. (4 marks)

Solution
$a(t) = pt - 13$ $v(t) = \frac{pt^2}{2} - 13t + c$ Since $v(0) = 5$, $c = 5$ $x(t) = \frac{pt^3}{6} - \frac{13t^2}{2} + 5t + k$ when $t = 2$: $-50 = \frac{8p}{6} - 16 + k$ when $t = 6$: $178 = 36p - 204 + k$ Solving gives: $p = 12$ and $k = -50$
Specific behaviours
✓ determines $v(t)$ and determines the constant c ✓ determines $x(t)$ ✓ finds the two displacement equations ✓ correctly determines p.

(c) When is the model train at rest? (2 marks)

Solution
$0 = 6t^2 - 13t + 5$ $t = \frac{1}{2}, \frac{5}{3}$ seconds
Specific behaviours
✓ equates velocity to zero ✓ solves to give both values of t.

(d) How far has the model train travelled when its acceleration is 47 cm/s^2? (2 marks)

Solution
$47 = 12t - 13$ $t = 5$ Dist travelled $= \int_0^5 \lvert v(t) \rvert dt$ $= \int_0^5 \lvert 6t^2 - 13t + 5 \rvert dt$ $= 115.7$ cm
Specific behaviours
✓ determines t when a = 47 cm/s/s ✓ calculates distance travelled

ACKNOWLEDGEMENTS

Questions 12(d), 18(b), 19(c)(i–iii)

Calculator screenshots: CASIO

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