Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS SPECIALIST

Calculator-free

ATAR course examination 2018

Marking Key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2018/62750

Section One: Calculator-free **35% (51 Marks)**

Question 1 **(5 marks)**

Functions *f, g* are defined such that:

$$f(x) = \sqrt{x-3}$$

$$g(x) = \frac{x}{x-2}$$

(a) Determine $g \circ f(x)$. (1 mark)

Solution
$g \circ f(x) = g\left(\sqrt{x-3}\right) = \dfrac{\sqrt{x-3}}{\sqrt{x-3} - 2}$
Specific behaviours
✓ forms a correct expression for $g \circ f(x)$

(b) Determine the domain for $g \circ f(x)$. (2 marks)

Solution
We require $x-3 \geq 0$ so the square root operation is defined. i.e. $x \geq 3$ However, we require that $\sqrt{x-3}-2 \neq 0$ so that division by zero does not occur. i.e. $\sqrt{x-3} \neq 2$ i.e. $x \neq 7$ Hence $D_{gof} = \{x \mid x \geq 3, x \neq 7\}$.
Specific behaviours
✓ states that $x \geq 3$ ✓ states that $x \neq 7$

(c) Given that $f^{-1}(x) = x^2 + 3$, is it true that $f^{-1}(-1) = 4$? (2 marks)

Explain.

Solution
No, this is FALSE. The domain for $f^{-1}(x)$ is $x \geq 0$ since $D_{f^{-1}} = R_f$. Hence $f^{-1}(-1)$ is not defined.
Specific behaviours
✓ states that the statement is false ✓ states that $f^{-1}(-1)$ is not defined

Question 2 **(6 marks)**

(a) Solve the following system of equations: (3 marks)

$4x - y - 2z = 5 \quad \ldots (1)$

$2x + y - z = 4 \quad \ldots (2)$

$x - y - z = 3 \quad \ldots (3)$

Solution
$(1) - 2(2): \quad -3y = -3$ $\therefore \quad y = 1$ $(1) - 4(3): \quad 3y + 2z = -7$ $\therefore \quad 3(1) + 2z = -7$ $\therefore \quad z = -5$ $(1): \quad 4x - (1) - 2(-5) = 5$ $\therefore \quad 4x + 9 = 5$ $\therefore \quad x = -1$ Hence the solution is unique: $x = -1, \quad y = 1, \quad z = -5$.
Specific behaviours
✓ uses appropriate alegbra correctly with **two** pairs of equations ✓ solves correctly to find the first variable ✓ solves correctly to find the second and third variables

Alternative Solution
$\begin{bmatrix} 4 & -1 & -2 & 5 \\ 2 & 1 & -1 & 4 \\ 1 & -1 & -1 & 3 \end{bmatrix} \begin{matrix} R_2 \to R_1 - 2R_2 \\ R_3 \to R_1 - 4R_3 \end{matrix} \rightarrow \begin{bmatrix} 4 & -1 & -2 & 5 \\ 0 & -3 & 0 & -3 \\ 0 & 3 & 2 & -7 \end{bmatrix}$ $\therefore \quad -3y = -3$ i.e. $y = 1$ $\therefore \quad 3y + 2z = -7$ i.e. $3(1) + 2z = -7$ i.e. $z = -5$ $\therefore \quad 4x - (1) - 2(-5) = 5$ i.e. $4x = -4$ i.e. $x = -1$ Hence the solution is unique: $x = -1, \quad y = 1, \quad z = -5$.
Specific behaviours
✓ applies at least **two** correct row operations ✓ solves correctly to find the first variable ✓ solves correctly to find the second and third variables

Question 2 (continued)

Consider another set of equations where k is a constant.

$$2x - y - z = 0$$
$$x - 2y - z = 2$$
$$x - 2y + kz = 6$$

It can be shown that this system of equations can be reduced to the following:

$$x = \frac{-2(k-1)}{3(k+1)}$$
$$y = \frac{-4(k+2)}{3(k+1)}$$
$$z = \frac{4}{k+1}$$

(b) Explain whether this system of equations will have a unique solution for all real values of k. If not, explain the geometric interpretation of this. (3 marks)

Solution
This system of equations will have a unique solution for all values of k provided $k \neq -1$. When $k = -1$ there will be NO solution. This is due to there being TWO planes that are PARALLEL to each other (equations 2 and 3).
Specific behaviours
✓ states there is a unique solution for $k \neq -1$ ✓ states that $k = -1$ will yield no solution ✓ states the geometric interpretation for $k = -1$ i.e. two planes are parallel

Question 3 **(9 marks)**

(a) Let $z = a + bi$ be any complex number. (3 marks)

Obtain an equation relating a, b given that $\text{Re}\left(\frac{z-i}{z}\right) = 0$.

Solution
$\frac{z-i}{z} = \frac{a+(b-1)i}{a+bi} = \frac{a+(b-1)i}{a+bi} \times \frac{a-bi}{a-bi}$ .... (1) $= \frac{a^2+b(b-1) - ai}{a^2+b^2}$ .... (2) Hence if the real part is ZERO then it must be true that: $a^2+b(b-1)=0$.   i.e. $a^2+b^2-b=0$
Specific behaviours
✓ forms the correct expression equivalent to (1) ✓ forms the correct expression equivalent to (2) ✓ forms the equation relating a, b stating the real part is zero

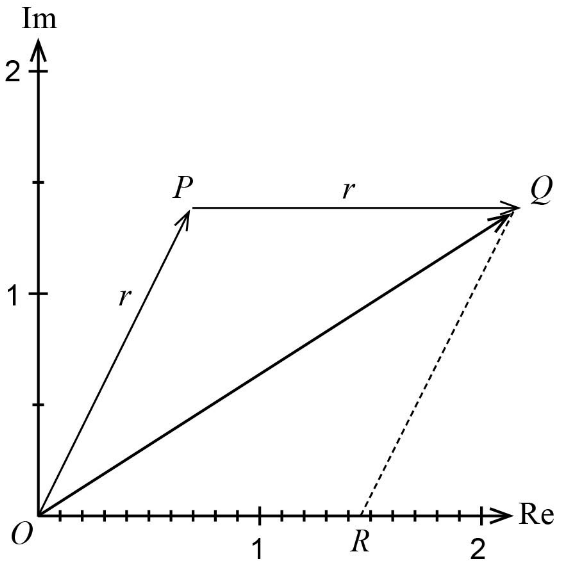
(b) Let $z = rcis\theta$ be any complex number. Obtain an expression for:

(i) $\frac{2i}{\bar{z}}$ in terms of r, θ. (3 marks)

Solution
$\frac{2i}{\bar{z}} = \frac{2cis\left(\frac{\pi}{2}\right)}{rcis(-\theta)} = \frac{2}{r}cis\left(\theta + \frac{\pi}{2}\right)$
Specific behaviours
✓ converts $2i$ into polar form correctly   ✓ writes the correct expression for $\bar{z}$ in polar form   ✓ divides polar forms correctly in terms of r, θ

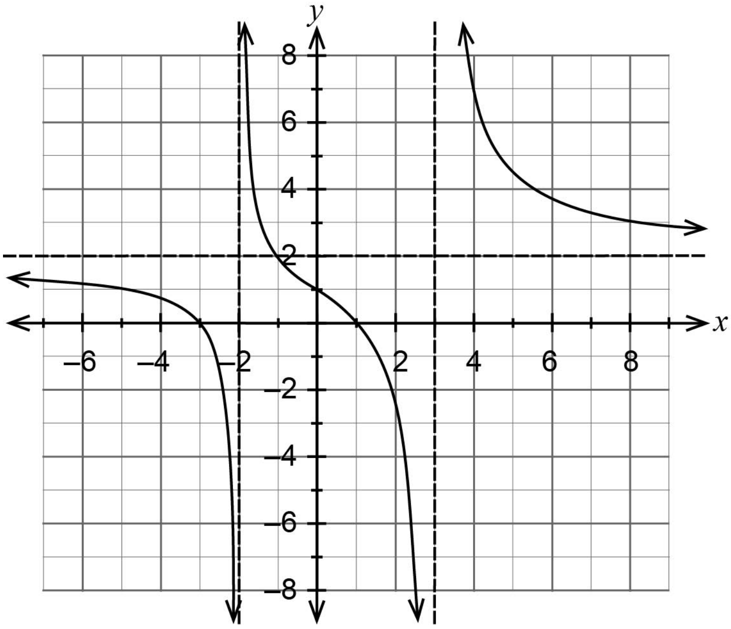
Question 3 (continued)

(ii) $arg(z+r)$ in terms of θ. (3 marks)

Solution

Let $z = \overrightarrow{OP} = rcis\theta$ and $r = \overrightarrow{PQ}$. $OPQR$ is a rhombus with side length r. Then the complex number $z + r = \overrightarrow{OQ}$ is a diagonal in the rhombus. It is a property that a diagonal bisects the angles in a rhombus. $\therefore \ s\angle QOR = \frac{1}{2} s\angle POR$ i.e. $\arg(z+r) = \frac{\theta}{2}$
Specific behaviours
✓ indicates the vector position for $z+r$ correctly ✓ identifies $z+r$ as the diagonal of a rhombus ✓ writes the correct expression for $arg(z+r)$

Question 4 **(4 marks)**

The graph of $f(x) = \dfrac{k(x-a)(x-b)}{(x-c)(x-d)}$ is shown below.



Determine the value of the constants a, b, c, d and k.

a	b	c	d	k
-3	1	-2	3	2

Explain your choice for the value of k.

Solution
Horizontal intercepts are $x=-3,\ x=1 \quad \therefore \quad a=-3,\ \ b=1$ Vertical asymptotes are $x=-2,\ x=3 \quad \therefore \quad c=-2,\ \ d=3$ Horizontal asymptote is $y=2 \quad \therefore \quad k=2$
Specific behaviours
✓ states the values for a, b correctly ✓ states the values for c, d correctly ✓ states the values for k correctly ✓ explains/justifies the value for k

Question 5 **(4 marks)**

Using the substitution $u = \cos(2x)$, evaluate exactly the definite integral

$$\int_0^{\frac{\pi}{4}} \cos^{1008}(2x)\sin(2x)\,dx\,.$$

Solution
$Put \quad u = \cos(2x) \quad \therefore \quad \frac{du}{dx} = -2\sin(2x) \quad i.e. \quad dx = -\frac{du}{2\sin(2x)}$
When $x = 0,\ u = 1$ and $x = \frac{\pi}{4},\ u = 0$
$\int_0^{\frac{\pi}{4}} \cos^{1008}(2x)\sin(2x)\,dx = \int_1^0 -\frac{u^{1008}}{2}\,du$
$= -\frac{1}{2}\left[\frac{u^{1009}}{1009}\right]_1^0 = -\frac{1}{2}\left[\frac{0}{1009} - \frac{1}{1009}\right] = \frac{1}{2018}$
Specific behaviours
✓ uses the substitution $u = \cos(2x)$ to express the integrand correctly in terms of u
✓ changes the limits correctly
✓ anti-differentiates correctly
✓ evaluates correctly

OR

Alternative Solution
$\int_0^{\frac{\pi}{4}} \cos^{1008}(2x)\sin(2x)\,dx = -\frac{1}{2}\int_0^{\frac{\pi}{4}} -2\sin(2x)\cos^{1008}(2x)\,dx$
$= -\frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{d}{dx}\left(\cos(2x)\right)\left(\cos(2x)\right)^{1008}dx$
$= -\frac{1}{2}\left[\frac{\cos^{1009}(2x)}{1009}\right]_0^{\frac{\pi}{4}}$
$= -\frac{1}{2018}\left[\cos^{1009}\left(\frac{\pi}{2}\right) - \cos^{1009}(0)\right]$
$= -\frac{1}{2018}[0-1] = \frac{1}{2018}$
Specific behaviours
✓ identifies $\sin(2x)$ as part of the derivative of $\cos(2x)$
✓ uses the factor $-\frac{1}{2}$ to express the definite integral
✓ anti-differentiates correctly using the next highest power
✓ evaluates correctly

Question 6 **(5 marks)**

(a) Given that $\dfrac{2}{(x+1)(x-1)} = \dfrac{a}{x-1} + \dfrac{b}{x+1}$, determine the values for a and b. (2 marks)

Solution
$\dfrac{a}{x-1} + \dfrac{b}{x+1} = \dfrac{a(x+1) + b(x-1)}{(x+1)(x-1)} = \dfrac{(a+b)x + (a-b)}{(x+1)(x-1)}$ Hence equating co-efficients we obtain $a+b=0$, $a-b=2$ i.e. $a=1,\ b=-1$.
Specific behaviours
✓ obtains the numerator correctly in simplifying the algebraic fractions   ✓ determines the values for a and b correctly

(b) Hence determine $\int \dfrac{1}{x^2-1}\,dx$. (3 marks)

Solution
$\int \dfrac{1}{x^2-1}\,dx = \dfrac{1}{2}\int \dfrac{2}{(x+1)(x-1)}\,dx$ $= \dfrac{1}{2}\int \left(\dfrac{1}{x-1} - \dfrac{1}{x+1}\right) dx$ $= \dfrac{1}{2}\left[\ln\lvert x-1\rvert - \ln\lvert x+1\rvert\right] + c$ $= \dfrac{1}{2}\ln\left\lvert\dfrac{x-1}{x+1}\right\rvert + c$
Specific behaviours
✓ writes the integral in terms of the partial fractions correctly   ✓ anti-differentiates correctly   ✓ uses an integration constant

Question 7 **(6 marks)**

(a) Solve the equation $z^3+1=0$ giving solutions in polar form $rcis\theta$. (3 marks)

Solution
$z^3 = -1 = cis(\pi)$ $\therefore\ z = cis\left(\dfrac{\pi+2\pi k}{3}\right)$ where $k=0,1,2$ $\therefore\ z_0 = cis\left(\dfrac{\pi}{3}\right)$, $z_1 = cis(\pi) = -1$, $z_2 = cis\left(\dfrac{5\pi}{3}\right) = cis\left(-\dfrac{\pi}{3}\right)$ i.e. $z = cis\left(\dfrac{\pi}{3}\right),\ cis(\pi),\ cis\left(-\dfrac{\pi}{3}\right)$
Specific behaviours
✓ expresses z^3 as $cis(\pi)$ ✓ states the first root in correct polar form for $k=0$ ✓ states the other 2 roots correctly using the argument separation $\dfrac{2\pi}{3}$

It can be shown that $P(z)=z^5-2z^4+5z^3+z^2-2z+5$ can be written in the form

$P(z)=(z^3+1)Q(z)$.

(b) Determine $Q(z)$. (1 mark)

Solution
$P(z) = z^5-2z^4+5z^3+z^2-2z+5$ $= z^3(z^2-2z+5) + 1(z^2-2z+5)$ $= (z^3+1)(z^2-2z+5)$ Hence $Q(z)=z^2-2z+5$.
Specific behaviours
✓ determines $Q(z)$ correctly

(c) Hence solve the equation $z^5 - 2z^4 + 5z^3 + z^2 - 2z + 5 = 0$ giving all solutions in Cartesian form $a + bi$. (2 marks)

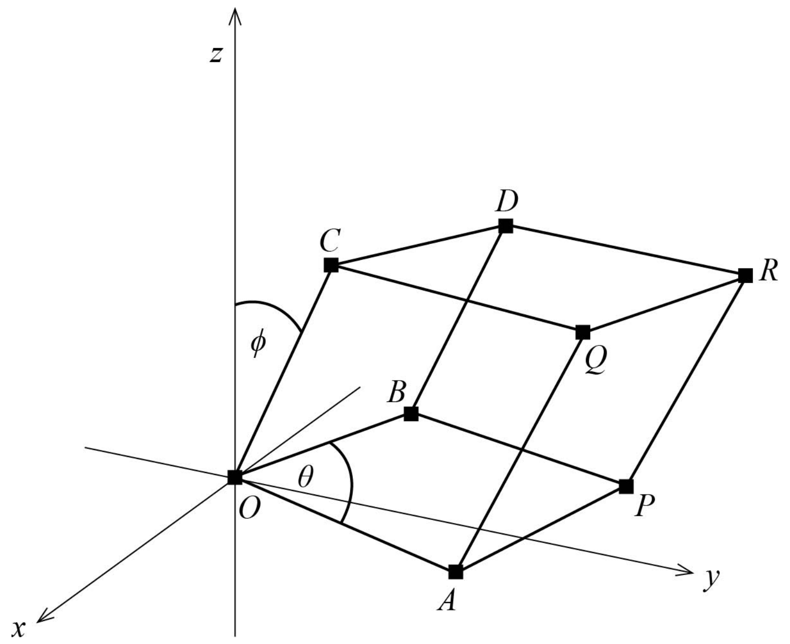
Solution
$\therefore \quad (z^2 - 2z + 5)(z^3 + 1) = 0$ i.e. Solve $z^2 - 2z + 5 = 0$ or $z^3 + 1 = 0$
i.e. $(z-1)^2 + 4 = 0$ or $z^3 = -1$
i.e. $[(z-1) + 2i][(z-1) - 2i] = 0$ or $z = -1,\ cis\left(\frac{\pi}{3}\right),\ cis\left(-\frac{\pi}{3}\right)$
i.e. $z = 1 + 2i,\ 1 - 2i,\ -1,\ \frac{1}{2} + \frac{\sqrt{3}}{2}i,\ \frac{1}{2} - \frac{\sqrt{3}}{2}i$
i.e. $z = 1 \pm 2i,\ -1,\ \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$
Specific behaviours
✓ states $z = 1 \pm 2i$ as solutions
✓ states $z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ as solutions

Question 8 **(5 marks)**

A parallelepiped is a prism where each face is a parallelogram. Let $OAPB$ be the parallelogram formed by the horizontal sides $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$ where

$$\underset{\sim}{a} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} \text{ and } \underset{\sim}{b} = \begin{pmatrix} -8 \\ 2 \\ 0 \end{pmatrix}.$$

The third side that forms the parallelepiped is $\underset{\sim}{c} = \overrightarrow{OC}$ where $\underset{\sim}{c} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$.



Let θ = the size of $\angle AOB$

ϕ = the angle between $\overrightarrow{OC}$ and the positive z axis

(a) Determine $\underset{\sim}{a} \times \underset{\sim}{b}$. (2 marks)

Solution
$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} \times \begin{pmatrix} -8 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6(0)-0(2) \\ 0(-8)-3(0) \\ 3(2)-6(-8) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 54 \end{pmatrix}$
Specific behaviours
✓ uses the correct form for each component for the cross product ✓ evaluates each component correctly

The volume of any prism can be found by considering the formula $Volume = Area(Base) \times h$, where $h =$ the perpendicular height of the prism.

It is also true that $|\underset{\sim}{a} \times \underset{\sim}{b}| = |\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta$.

(b) Explain why $\underset{\sim}{c} \bullet (\underset{\sim}{a} \times \underset{\sim}{b})$ will determine the volume of the parallelepiped. (2 marks)

Solution
From $Volume = Area(Base) \times h$ Area($OAPB$) $= 2 \times \frac{1}{2}(OA)(OB)\sin\theta = |\underset{\sim}{a} \times \underset{\sim}{b}|$ Perpendicular height $h = (OC)\cos\phi$ Note that $\underset{\sim}{a} \times \underset{\sim}{b}$ is parallel to the z-axis hence the angle between $\underset{\sim}{c}$ and $\underset{\sim}{a} \times \underset{\sim}{b}$ is also ϕ. Hence volume $V = |\underset{\sim}{c}|\cos\phi \ |\underset{\sim}{a} \times \underset{\sim}{b}| = |\underset{\sim}{c}||\underset{\sim}{a} \times \underset{\sim}{b}|\cos\phi$. i.e. a dot product $= \underset{\sim}{c} \bullet (\underset{\sim}{a} \times \underset{\sim}{b})$. Since the value $\cos\phi$ may be less than zero we consider the absolute value of this dot product. i.e. $V = |\underset{\sim}{c} \bullet (\underset{\sim}{a} \times \underset{\sim}{b})|$
Specific behaviours
✓ justifies that $|\underset{\sim}{a} \times \underset{\sim}{b}|$ determines the area of the parallelogram base ✓ justifies that the perpendicular height $h = |\underset{\sim}{c}|\cos\phi$

(c) Hence determine the exact volume of the parallelepiped. (1 mark)

Solution
Using $V = \underset{\sim}{c} \bullet (\underset{\sim}{a} \times \underset{\sim}{b}) = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 0 \\ 54 \end{pmatrix} = (-1)0 + 2(0) + 5(54) = 270 \textit{ cubic units}$
Specific behaviours
✓ evaluates the dot product correctly

Question 9 **(7 marks)**

(a) By using an appropriate trigonometric identity, simplify in terms of u, the expression $x^2 - 2x + 4$ where $x = \sqrt{3}\tan(u) + 1$. (2 marks)

Solution
$x^2 - 2x + 4 = \left(\sqrt{3}\tan u + 1\right)^2 - 2\left(\sqrt{3}\tan u + 1\right) + 4$ $= 3\tan^2 u + 2\sqrt{3}\tan u + 1 - 2\sqrt{3}\tan u - 2 + 4$ $= 3\tan^2 u + 3$ $= 3\left(\tan^2 u + 1\right)$ $= 3\sec^2 u$
Specific behaviours
✓ expands correctly to obtain $3\tan^2 u + 3$ ✓ uses the trigonometric identity correctly to simplify to $3\sec^2 u$

(b) Hence evaluate $\int_1^2 \frac{dx}{\left(x^2 - 2x + 4\right)^{\frac{3}{2}}}$ exactly. (5 marks)

Solution
Using $x = \sqrt{3}\tan(u) + 1 \quad \therefore \quad \frac{dx}{du} = \sqrt{3}\sec^2 u$ *i.e.* $dx = \sqrt{3}\sec^2 u\,.du$ When $x = 1,\ u = 0$ and $x = 2,\ u = \frac{\pi}{6}$ $\therefore \int_1^2 \frac{dx}{\left(x^2 - 2x + 4\right)^{\frac{3}{2}}} = \int_0^{\frac{\pi}{6}} \frac{\sqrt{3}\sec^2 u\,.du}{\left(3\sec^2 u\right)^{\frac{3}{2}}} = \int_0^{\frac{\pi}{6}} \frac{\sqrt{3}\sec^2 u\,.du}{3\sqrt{3}\sec^3 u} = \int_0^{\frac{\pi}{6}} \frac{du}{3\sec u}$ $= \int_0^{\frac{\pi}{6}} \frac{\cos u}{3}\,du$ $= \left[\frac{\sin u}{3}\right]_0^{\frac{\pi}{6}} = \frac{1}{3}\left(\frac{1}{2}\right) - \frac{1}{3}(0) = \frac{1}{6}$
Specific behaviours
✓ obtains dx correctly in terms of du ✓ changes the limits correctly in terms of u ✓ simplifies the integrand correctly in terms of u using the expression from part (a)   ✓ anti-differentiates correctly   ✓ evaluates correctly

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