



MATHEMATICS METHODS

Calculator-assumed

ATAR course examination 2021

Marking key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

Section Two: Calculator-assumed

65% (100 Marks)

Question 8

(9 marks)

The weights W (in grams) of carrots sold at a supermarket have been found to be normally distributed with a mean of 142.8 g and a standard deviation of 30.6 g.

- (a) Determine the percentage of carrots sold at the supermarket that weigh more than 155 g. (2 marks)

Solution
$P(W > 155) = 0.3451$ $0.3451 \times 100 = 34.51\%$
Specific behaviours
✓ obtains correct probability ✓ obtains correct percentage

Carrots sold at the supermarket are classified by weight, as shown in the table below.

Classification	Small	Medium	Large	Extra large
Weight W (grams)	$W \leq 110$	$110 < W \leq 155$	$155 < W \leq 210$	$W > 210$
$P(W)$	0.1418	0.5131	0.3310	0.3451 – 0.3310 = 0.0141

- (b) Complete the table above, providing the missing probabilities. (2 marks)

Solution
See table above
Specific behaviours
✓ determines one correct probability ✓ determines second correct probability

- (c) Of the carrots being sold at the supermarket that are **not** of medium weight, what proportion is small? (2 marks)

Solution
$P(\text{Small} \mid \text{not Medium}) = \frac{P(\text{Small})}{P(\text{not Medium})} = \frac{0.1418}{0.4869} = 0.2912$
Specific behaviours
✓ determines correct denominator ✓ determines correct numerator and obtains final answer

The supermarket sells bags of mixed-weight carrots, with 12 randomly-selected carrots placed in each bag.

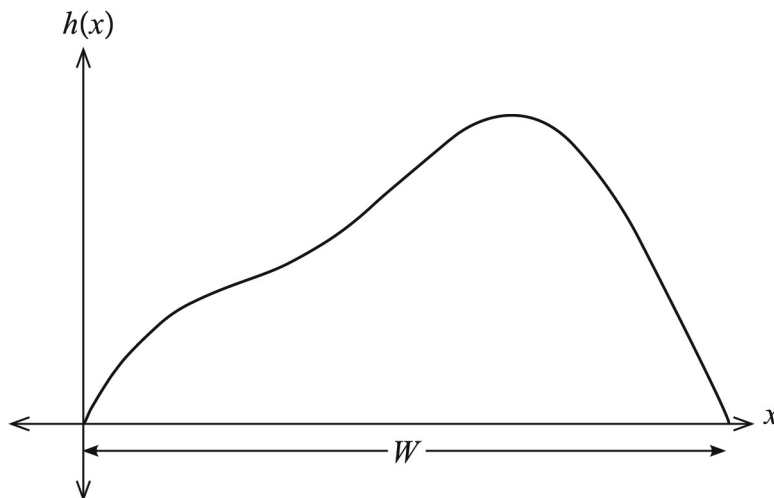
- (d) If a customer purchases a bag of mixed-weight carrots, determine the probability that there will be at most two small carrots in the bag. (3 marks)

Solution
Let the random variable Y denote the number of small carrots in a bag. Then $Y \sim \text{Bin}(12, 0.1418)$ We need $P(Y \leq 2) = 0.7637$
Specific behaviours
<ul style="list-style-type: none">✓ defines appropriate random variable and states the correct binomial distribution✓ states the correct probability statement✓ computes the probability

Question 9

(8 marks)

The Interesting Architecture company has designed a building with a uniform cross-section shown in the figure below.



With reference to the figure, the height $h(x)$ of the building at a point x along its width is given by

$$h(x) = 4 \sin\left(x - \frac{3\pi}{2}\right) - x^2 + 3\pi x - 4, \text{ where } h \text{ and } 0 \leq x \leq W \text{ are measured in metres.}$$

- (a) Determine the width W of the building to the nearest centimetre. (2 marks)

Solution
$h(W) = 0$ $W = 8.64 \text{ m (or 864 cm).}$
Specific behaviours
✓ sets $h(W) = 0$ ✓ solves for W

- (b) Determine $h'(x)$. (1 mark)

Solution
$h'(x) = 4 \cos\left(x - \frac{3\pi}{2}\right) - 2x + 3\pi$
Specific behaviours
✓ differentiates $h(x)$

- (c) Determine, to the nearest centimetre, the value of x at which the height of the building is maximum and state this maximum height. (2 marks)

Solution
Setting $h'(x) = 0$ gives $x = 5.74$ m.
Hence the maximum height $h(5.74) = 20.57$ m.
Specific behaviours
✓ sets $h'(x) = 0$ and solves it to obtain the value of x for maximum height
✓ states the maximum height

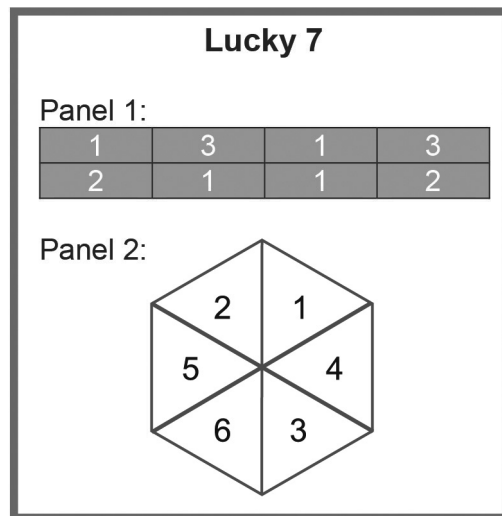
- (d) An adventure company allows tourists to climb from the ground on the left of the building, then along the outside of the building to the top. The company installs a platform that allows climbers to rest on their way up to the top. The platform is located on the second half of the climb, at the point where it is the steepest. How high off the ground, to the nearest centimetre, is it positioned? (3 marks)

Solution
The climb is steepest when the gradient is a maximum.
The second derivative is given by
$h''(x) = -4 \sin\left(x - \frac{3\pi}{2}\right) - 2$
$h''(x) = 0$ at the points where $x = 2.09$ and $x = 4.19$
When $x = 4.19$ (second half of the climb), the platform is 15.93 metres off the ground.
Specific behaviours
✓ obtains the second derivative
✓ equates $h''(x)$ to zero and determines both x values
✓ identifies correct point and determines the height of the platform off the ground

Question 10

(8 marks)

A charity organisation has printed 'Lucky 7' scratchie tickets as a fundraiser for use at two special events. The tickets contain two panels. Each ticket has the same numbers as the sample ticket shown below, arranged randomly and hidden within each panel.



A player scratches one section of each panel to reveal a number. The two numbers revealed are then added together. If the total is seven or higher, the player wins a prize.

At the first event, 400 tickets are purchased, and a prize is won on 124 occasions. Let p denote the probability that a prize is won.

- (a) Determine the sample proportion of times that a prize is won at the first event. (1 mark)

Solution
$\hat{p} = \frac{124}{400} = 0.31$
Specific behaviours
✓ correctly determines the sample proportion

- (b) Show that the probability p of winning a prize is $\frac{7}{24}$. (2 marks)

Solution		
score	combinations	probability
7	3,4 or 2,5 or 1,6	$\frac{2}{8} \times \frac{1}{6} + \frac{2}{8} \times \frac{1}{6} + \frac{4}{8} \times \frac{1}{6} = \frac{8}{48}$
8	3,5 or 2,6	$\frac{4}{48}$
9	3,6	$\frac{2}{48}$

Probability of a prize = $\frac{8}{48} + \frac{4}{48} + \frac{2}{48} = \frac{14}{48} = \frac{7}{24}$

Specific behaviours	
✓ shows how to determine at least one probability	
✓ correctly shows all three probabilities and shows they sum to $\frac{7}{24}$	

- (c) Calculate the mean and standard deviation of the sample proportion of times a prize is won when 400 tickets are purchased. (2 marks)

Solution	
mean = $p = \frac{7}{24} = 0.2917$	
standard deviation = $\sqrt{\frac{\frac{7}{24} \left(1 - \frac{7}{24}\right)}{400}} = \frac{\sqrt{119}}{480} = 0.02273$	

Specific behaviours	
✓ correctly determines the mean	
✓ correctly determines the standard deviation	

- (d) At a second event, 400 scratchie tickets are again purchased. If the sample proportion was 0.6 standard deviations from the population proportion, how many prizes were won at the second event? (3 marks)

Solution	
$ \text{Second } \hat{p} - p = 0.6 \times 0.02273 = 0.01364$	
Second $\hat{p} = 0.2917 \pm 0.01364$	
Possible number of prizes are: $400 \times (0.2917 \pm 0.01364) \approx 111$ or 122	

Specific behaviours	
✓ correctly determines the difference between the sample and population means	
✓ states the two possibilities for second event sample proportion	
✓ determines the possible number of prizes	

Question 11

(17 marks)

A new political party, the Sustainable Energy Party, is planning to have candidates run in the next election. Researchers have collected data that suggests the proportion of voters likely to vote for the party to be 23%.

One year before the next election, random samples of 400 voters were taken in a particular electorate. Let \hat{p} denote the sample proportion of voters who indicated they would vote for the Sustainable Energy Party at the next election.

- (a) State the distribution of \hat{p} . (3 marks)

Solution
$\hat{p} \sim N\left(0.23, \frac{0.23 \times 0.77}{400}\right)$ <p>that is,</p> $\hat{p} \sim N(0.23, 0.000\ 442\ 75)$
Specific behaviours
<ul style="list-style-type: none"> ✓ states the distribution is normal ✓ gives the correct mean ✓ gives the correct variance

- (b) Calculate the probability that the proportion of voters likely to vote for the Sustainable Energy Party in a sample of 400 is less than 0.20. (3 marks)

Solution
$P(\hat{p} < 0.20) = 0.076\ 97$
Specific behaviours
<ul style="list-style-type: none"> ✓ writes correct probability statement ✓ uses correct mean and standard deviation ✓ obtains final answer

One week before the election, researchers believed that the proportion of voters likely to vote for the party in that same electorate had increased. A random sample of 200 voters was taken at this time, and 55 of them indicated they would vote for the Sustainable Energy Party at the next election.

- (c) Based on this sample, estimate the proportion of voters likely to vote for the Sustainable Energy Party in this electorate. (1 mark)

Solution
$\hat{p} = \frac{55}{200} = 0.275$
Specific behaviours
<ul style="list-style-type: none"> ✓ calculates sample proportion correctly

- (d) For a 99% confidence interval, what is the margin of error of the sample proportion of voters likely to vote for the Sustainable Energy Party in this electorate, based on this sample? (2 marks)

Solution
$E = 2.576 \sqrt{\frac{0.275 \times 0.725}{200}} = 0.08133$
Specific behaviours
<ul style="list-style-type: none"> ✓ substitutes correct values in the formula for margin of error ✓ calculates margin of error correctly

- (e) Based on this sample, calculate a 95% confidence interval for the population proportion of voters likely to vote for the Sustainable Energy Party in this electorate. (3 marks)

Solution
$95\% \text{ CI} = \left(0.275 - 1.96 \times \sqrt{\frac{0.275(0.725)}{200}}, 0.275 + 1.96 \times \sqrt{\frac{0.275(0.725)}{200}} \right)$ $95\% \text{ CI} = (0.2131, 0.3369)$
Specific behaviours
<ul style="list-style-type: none"> ✓ uses the correct critical value from the normal distribution ✓ substitutes correct values in the expression for the confidence interval ✓ calculates the confidence interval correctly

- (f) Based on the research, did the proportion of voters likely to vote for the Sustainable Energy Party in this electorate increase in the year leading up to the election? Justify your answer. (2 marks)

Solution
The 95% confidence interval for the new sample (from part (e)) contains the value of the proportion for the earlier sample, so based on this we concluded that there is not enough evidence to determine whether the voters likely to vote for the Sustainable Energy party in this electorate has increased.
Specific behaviours
<ul style="list-style-type: none"> ✓ states that the confidence interval contains the proportion from the earlier sample ✓ concludes that there is not enough evidence to determine whether the proportion has increased

Question 11 (continued)

- (g) The analysis above models the number of voters likely to vote for the Sustainable Energy Party as binomially distributed. State and discuss the validity of any assumptions for the binomial distribution in this context. (3 marks)

Solution
1. Voters either vote for the party or not (success or failure). 2. The voters likely to vote for the Sustainable Energy party are independent of each other. This is a reasonable assumption. 3. The probability of a voter likely to vote for the Sustainable Energy party is the same for all voters. This is most likely not valid, as the probability may depend on other factors, such as the age of the voter, occupation, socio-economic status, employment status.
Specific behaviours
✓ states the first assumption with justification ✓ states the second assumption with justification ✓ states the third assumption with justification

Question 12

(15 marks)

Let $f(x) = x^2e^x$.

- (a) Show that $f'(x) = xe^x(x + 2)$. (2 marks)

Solution
$f'(x) = 2xe^x + x^2e^x = xe^x(x + 2)$
Specific behaviours
<ul style="list-style-type: none"> ✓ differentiates using product rule ✓ factorises correctly

- (b) Use calculus to determine all the stationary points of $f(x)$ and determine their nature. (7 marks)

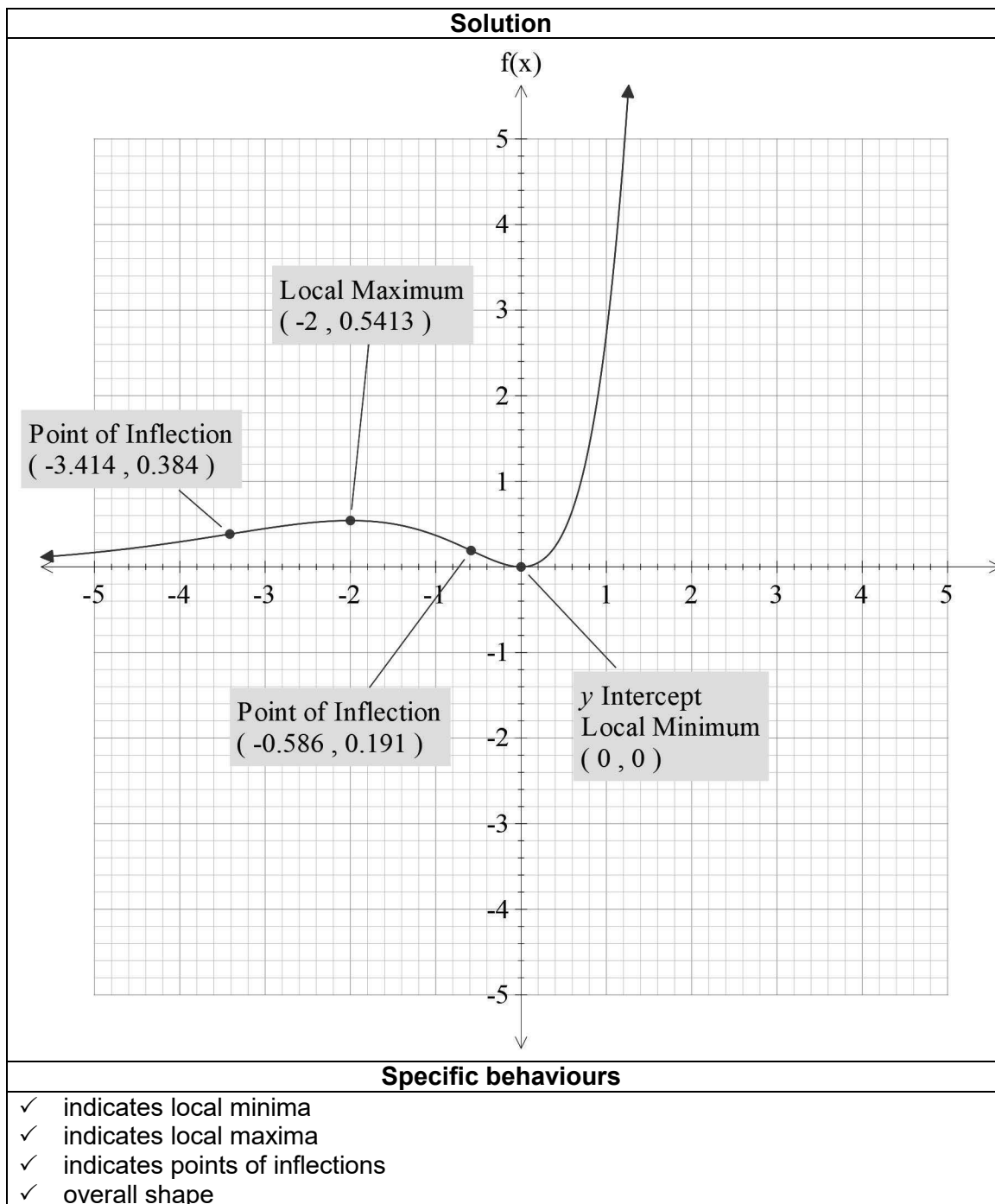
Solution
$f'(x) = 0$ $\Rightarrow xe^x(x + 2) = 0$ $\Rightarrow x = 0, -2$ $f''(x) = 2e^x + 2xe^x + 2xe^x + x^2e^x = e^x(2 + 4x + x^2)$ $f''(0) = 2 > 0 \Rightarrow$ Local minima $f(0) = 0$ $f''(-2) = -2e^{-2} < 0 \Rightarrow$ Local maxima $f(-2) = 4e^{-2} \approx 0.54$
Specific behaviours
<ul style="list-style-type: none"> ✓ sets first derivative equal to 0 ✓ obtains the two solutions ✓ finds the second derivative ✓ evaluates the second derivative at $x = 0$ and concludes it is a local minima ✓ obtains y coordinate at minima ✓ evaluates the second derivative at $x = -2$ and concludes it is a local maxima ✓ obtains y coordinate at maxima

- (c) Determine the coordinates of any points of inflection. (2 marks)

Solution
$f''(x) = 0$ $\Rightarrow e^x(2 + 4x + x^2) = 0$ $\Rightarrow x = -2 \pm \sqrt{2} \approx -3.4, -0.59$ \Rightarrow points are $(-3.4, 0.38)$ and $(-0.59, 0.19)$
Specific behaviours
<ul style="list-style-type: none"> ✓ sets second derivative equal to 0 ✓ obtains the two points

Question 12 (continued)

- (d) Hence sketch the graph of $f(x)$, clearly indicating the location of all stationary points and points of inflection. (4 marks)



Question 13

(14 marks)

A carnival game involves five buckets, each containing 5 blue balls and 15 red balls. A player blindly selects a ball from each bucket and wins the game if they select at least 4 blue balls. Let X denote the number of blue balls selected.

- (a) State the distribution of X , including its parameters. (2 marks)

Solution
$X \sim \text{Bin}(5, 0.25)$
Specific behaviours
<ul style="list-style-type: none"> ✓ recognises the distribution is binomial ✓ determines correct parameters

- (b) What is the probability of a player winning the game on any given attempt. (2 marks)

Solution
$P(X = 4) + P(X = 5) = \binom{5}{4} \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right) + \binom{5}{5} \left(\frac{1}{4}\right)^5$ $= \frac{15}{1024} + \frac{1}{1024}$ $= \frac{1}{64}$
Specific behaviours
<ul style="list-style-type: none"> ✓ states correct probability expression ✓ calculates correct probability

- (c) Players are charged \$2 for each attempt at the game and offered a \$150 prize if they win the game. By providing appropriate numerical justification, explain why this is not a good idea for the carnival organisers. (2 marks)

Solution
<p>The expected payout, E, per game is</p> $E = \frac{1}{64} \times \$150 = \$2.34$ <p>If the carnival organisers only charge \$2 per game then on average they will lose approximately 34c per game.</p>
Specific behaviours
<ul style="list-style-type: none"> ✓ determines expected payout per game ✓ concludes that charging less than the expected payout per game will lead to a loss of money on average

Question 13 (continued)

An observer records the outcome of 100 consecutive games and determines the 90% and 95% confidence intervals for the proportion of wins, p . The confidence intervals are (0.04, 0.16) and (0.05, 0.15).

- (d) Which of these intervals is the 95% confidence interval for p ? Justify your answer. (2 marks)

Solution
(0.04, 0.16) is the 95% confidence interval as it is the wider of the two intervals provided (the 95% confidence interval is wider than the 90% confidence interval).
Specific behaviours
<ul style="list-style-type: none"> ✓ chooses the correct interval ✓ provides correct justification for the choice

- (e) How many wins were observed out of the 100 games? (2 marks)

Solution
The mid-point of the confidence intervals gives $\hat{p} = 0.1$. Since 100 games were observed it means that $0.1 \times 100 = 10$ wins were observed.
Specific behaviours
<ul style="list-style-type: none"> ✓ determines the value of \hat{p} ✓ determines the number of wins observed

- (f) Determine what you would expect to happen to the width of the confidence intervals if 400 games had been observed. (2 marks)

Solution
The width of the confidence interval is proportional to $\sqrt{\frac{1}{n}}$. Hence increasing the number of observed games by a factor of 4 will lead to the confidence interval width reducing by a factor of 2 (i.e. halved).
Specific behaviours
<ul style="list-style-type: none"> ✓ states that the width will reduce ✓ determines that the reduction is by a factor of 2

- (g) The true proportion of wins does not lie within either of the above confidence intervals. Does this suggest that a sampling error was made? Justify your answer. (2 marks)

Solution
A mistake has not necessarily been made. Not all 90% or 95% confidence intervals will contain the true proportion p .
Specific behaviours
<ul style="list-style-type: none"> ✓ states that a mistake has not necessarily been made ✓ states that not all confidence intervals contain the true population proportion

Question 14

(5 marks)

The displacement in metres, $x(t)$, of a power boat t seconds after it was launched is given by:

$$x(t) = \frac{5t(t^2 - 15t + 48)}{6}, \quad t \geq 0$$

How far has the power boat travelled before its acceleration is zero?

Solution
$x(t) = \frac{5t(t^2 - 15t + 48)}{6}, \quad t \geq 0$
$v(t) = \frac{dx}{dt} = \frac{5t^2 - 50t + 80}{2}$
$a(t) = \frac{d^2x}{dt^2} = 5t - 25$
$5t - 25 = 0$
$\therefore t = 5$
$\text{Distance travelled} = \int_0^5 \left \frac{5t^2 - 50t + 80}{2} \right dt$
$= \frac{245}{3}$
$\approx 81.7 \text{ metres}$
Specific behaviours
<ul style="list-style-type: none">✓ determines an expression for velocity✓ determines an expression for acceleration✓ equates acceleration to zero and determines t✓ shows integration expression for distance travelled✓ determines how far the power boat has travelled

Question 15

(4 marks)

The graph of $y = m \log_3(x - p) + q$ has a vertical asymptote at $x = 5$.

(a) Explain why $p = 5$.

(2 marks)

Solution
The graph of $y = \log_3(x)$ has a vertical asymptote at $x = 0$. The graph of $y = m \log_3(x - p) + q$ has been translated p units to the right. Since this graph has a vertical asymptote at $x = 5$, p must equal 5.
Specific behaviours
✓ states the graph of $y = \log_3(x)$ has a vertical asymptote at $x = 0$ ✓ identifies a horizontal translation and equates vertical asymptote to value of p

(b) If this graph passes through the points $(6, 2)$ and $(14, -6)$, determine the values of m and q .

(2 marks)

Solution
Substituting the points into equation: $2 = m \log_3(1) + q$ (1) $-6 = m \log_3(9) + q$ (2) Equation (1) gives $q = 2$ Equation (2) gives: $-6 = m \log_3(3^2) + 2$ $-8 = 2m$ $m = -4$
Specific behaviours
✓ substitutes the points into the equation ✓ determines the values of q and m

Question 16

(12 marks)

An analyst was hired by a large company at the beginning of 2021 to develop a model to predict profit. At that time, the company’s profit was \$4 million. The model developed by the analyst was:

$$P(x) = \frac{20\ln(x+a)}{x+5}$$

where $P(x)$ is the profit in millions of dollars after x weeks and a is a constant.

- (a) Show that $a = e$. (2 marks)

Solution
Since $P(0) = 4$, we require $\ln(a) = 1$, giving $a = e$.
Specific behaviours
✓ recognises $P(0) = 4$
✓ obtains $\ln(a) = 1$

- (b) What does the model predict the profit will be after five weeks? (1 mark)

Solution
$P(5) = \frac{20\ln(5+e)}{5+5}$ $= 4.087 \text{ (3 d.p.)}$ Profit will be approximately \$4 087 000
Specific behaviours
✓ states the profit

- (c) Showing use of the quotient rule, determine an equation that, when solved, will give the time when the model predicts the profit will be maximised. (3 marks)

Solution
$P'(x) = \frac{(x+5)\frac{20}{(x+e)} - 20\ln(x+e)}{(x+5)^2}$ For maximum profit we require :
$\frac{(x+5)\frac{20}{(x+e)} - 20\ln(x+e)}{(x+5)^2} = 0$
Specific behaviours
✓ demonstrates use of the quotient rule ✓ writes correct expression for $P'(x)$ ✓ equates $P'(x)$ to zero

Question 16 (continued)

- (d) What is this maximum profit and during which week will it occur? (2 marks)

Solution
Maximum profit is approximately \$4 436 000 This occurs when $x \approx 1.79$, so during the second week.
Specific behaviours
✓ states maximum profit ✓ states it occurs in the second week

- (e) According to the model, during which week will the company's profit fall below its value at the beginning of 2021? (1 mark)

Solution
$4 = \frac{20 \ln(x + e)}{x + 5}$ $x = 0$ or 5.581 The model predicts during the 6 th week
Specific behaviours
✓ determines when the profit falls below the 2021 value

The model proved accurate and after 10 weeks the company implemented some changes. From this time the analyst used a new model to predict the profit:

$$N(y) = 2e^{b(10+y)},$$

where $N(y)$ is the profit in millions of dollars y weeks from this point in time and b is a constant.

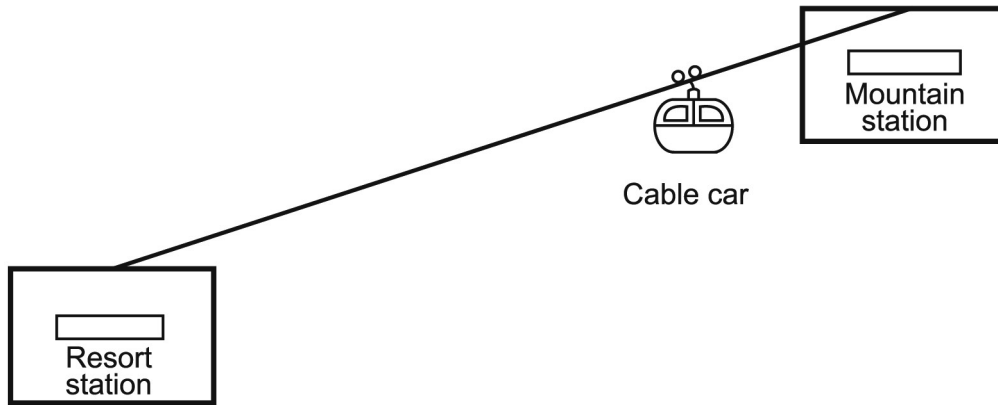
- (f) The company is projecting its profit to exceed \$5 million. During which week does the new model suggest this will happen? (3 marks)

Solution
$P(10) = 3.39072 = N(0)$ $3.39072 = 2e^{10b}$ $b = 0.05279$ $5 = 2e^{b(10+y)}$ $y \approx 7.36$ Profit should exceed \$5 million during the 8 th week after the changes.
Specific behaviours
✓ determines $P(10)$ ✓ determines the value of the constant b ✓ determines the week when the profit exceeds \$5 million

Question 17

(8 marks)

A resort in the Swiss Alps features a cable car that travels from the resort station to the mountain station. Engineers are fixing a cable car that unexpectedly stopped shortly before it reached the mountain station. The engineers are ready to test the cable car. For the purposes of the test the cable car will initially be at rest in its current position, will head up the mountain, stop at the mountain station and immediately return to the resort station where it will stop, and the test will be complete.



The test begins and engineers believe that the acceleration, $a(t)$, of the cable car during the test will be: $a(t) = kt^2 - 23t + 20k$, measured in m/min^2 . The variable t is the number of minutes from the moment the cable car leaves its position and k is a constant. After two minutes, the engineers expect that the cable car will be travelling with velocity $18 \text{ m}/\text{min}$ and will not yet have reached the mountain station.

- (a) Determine the value of the constant k . (3 marks)

Solution
$a(t) = kt^2 - 23t + 20k$ $v(t) = \frac{kt^3}{3} - \frac{23t^2}{2} + 20kt + c$ $c = 0 \text{ since } v(0) = 0$ $v(2) = 18$ $18 = \frac{8k}{3} - 46 + 40k$ $\therefore k = 1.5$
Specific behaviours
<ul style="list-style-type: none"> ✓ determines an expression for the velocity including determining $c = 0$ ✓ substitutes $t = 2, v = 18$ ✓ correctly determines k

Question 17 (continued)

- (b) Once the cable car leaves the mountain station, how long should it take to return to the resort station? (3 marks)

Solution
$v(t) = 0.5t^3 - 11.5t^2 + 30t$ <p>Cable Car stops at the resort station \Rightarrow velocity = 0</p> $0 = 0.5t^3 - 11.5t^2 + 30t$ $\therefore t = 0, 3, 20$ <p>It takes $20 - 3 = 17$ minutes to reach the resort station.</p>
Specific behaviours
<ul style="list-style-type: none"> ✓ equates the velocity to zero ✓ solves for t ✓ states the time taken

- (c) Unfortunately, 10 minutes into the test, the cable car breaks down again. According to the engineers' model, how far is the cable car from the mountain station at this time? (2 marks)

Solution
$\text{dist travelled} = \int_3^{10} 0.5t^3 - 11.5t^2 + 30t dt$ $= 1124.958$ $\approx 1125 \text{ metres}$ <p>The cable car is 1125 metres below the mountain station.</p>
Specific behaviours
<ul style="list-style-type: none"> ✓ writes an expression that can be used to determine position ✓ determines the position of the cable car

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