Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS SPECIALIST

Calculator-assumed

ATAR course examination 2021

Marking key

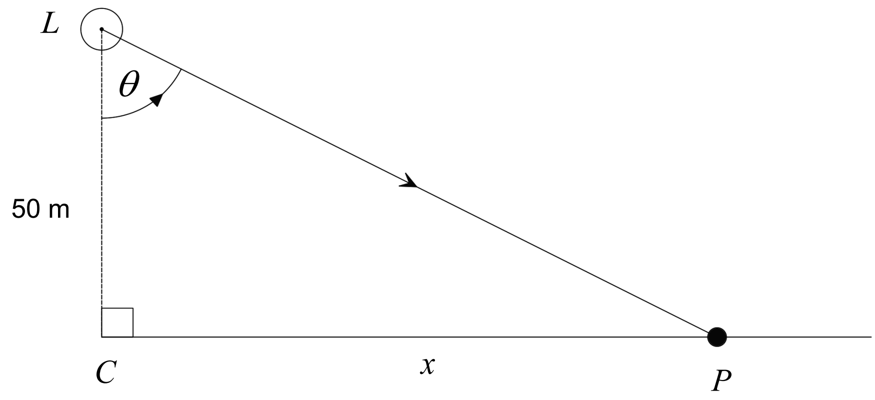
Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2022/2172
Web version of 2021/67369

Section Two: Calculator-assumed **65% (92 Marks)**

Question 9 **(5 marks)**

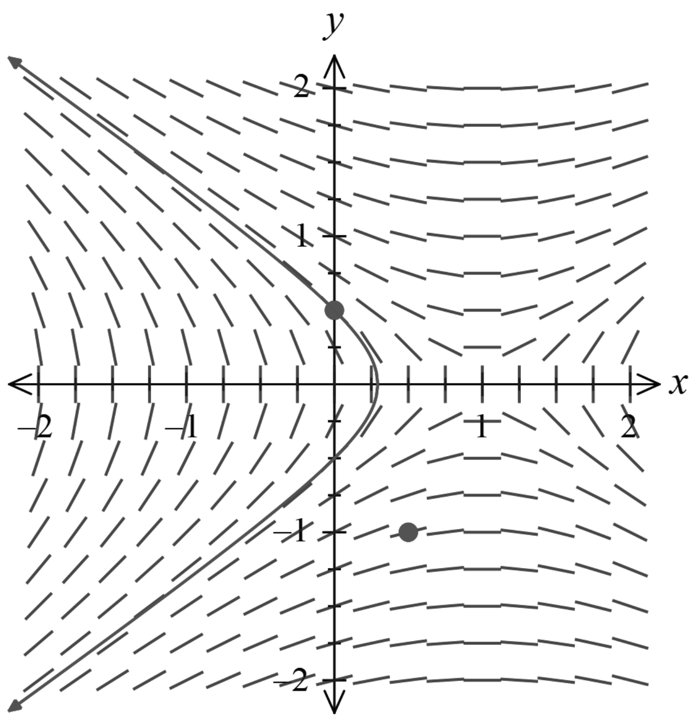
A beam of light completes three revolutions each minute from a lighthouse L that is 50 metres from a coastline. Determine the speed of the beam of light moving along the coast when it is at point P, 100 metres up the coast, correct to the nearest 0.01 metres per second.



Solution
$\frac{d\theta}{dt} = \frac{3\times 2\pi}{60} = \frac{\pi}{10}$ radians per second (0.314159…) In right ΔLCP: $\tan\theta = \frac{x}{50}$ $\therefore$ $x = 50\tan\theta$ When $x=100$ $\tan\theta = 2$ i.e. $\sec\theta=\sqrt{5}$ i.e. $\theta = 1.1071..$ $\therefore$ $\frac{dx}{dt} = 50\sec^2\theta \times \frac{d\theta}{dt}$ $= 50\left(\sqrt{5}\right)^2\left(\frac{\pi}{10}\right)$ $= 25\pi$ $m/\sec$ $= 78.5398...$ $m/\sec$ $\therefore$ The beam is moving at a speed of 78.54 metres per second at point P.
Specific behaviours
✓ determines the angular rate of change correctly ✓ forms the correct expression for the distance x in terms of θ ✓ differentiates correctly to relate the rates of change ✓ substitutes correctly the value for $\sec\theta$, $\cos\theta$ *or* θ ✓ calculates the speed correctly to 0.01 (and states the correct units)

Question 10 **(8 marks)**

Part of the slope field given by $\frac{dy}{dx} = \frac{x-1}{2y}$ is shown below.



(a) Calculate and draw the slope field at the point $(0.5, -1)$. (3 marks)

Solution
At $(0.5, -1)$, $\frac{dy}{dx} = \frac{(0.5)-1}{2(-1)} = 0.25$
Specific behaviours
✓ substitutes correctly into the expression for $\frac{dy}{dx}$ ✓ calculates the slope field correctly ✓ indicates the correct slope orientation at $(0.5, -1)$

Question 10 (continued)

(b) Determine the equation of the solution curve that contains the point $(0,\ 0.5)$. (3 marks)

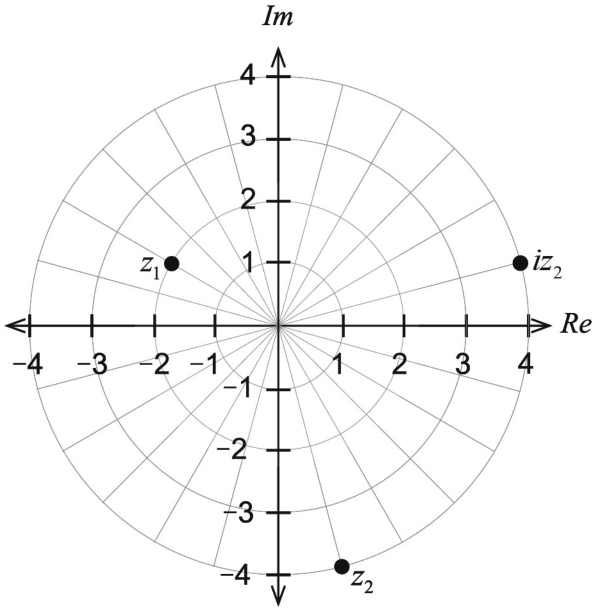
Solution
From $\dfrac{dy}{dx} = \dfrac{x-1}{2y}$ we obtain $\int 2y\,dy = \int (x-1)\,dx$ i.e. $y^2 = \dfrac{x^2}{2} - x + c$ Using $(0,\ 0.5)$, $(0.5)^2 = \dfrac{0^2}{2} - 0 + c$ $\therefore\ c = 0.25$ $\therefore\ y^2 = \dfrac{x^2}{2} - x + \dfrac{1}{4}$ OR $y^2 = \dfrac{(x-1)^2}{2} + c$ Using $(0,\ 0.5)$, $(0.5)^2 = \dfrac{(0-1)^2}{2} + c$ $\therefore\ c = -0.25$ $\therefore\ y^2 = \dfrac{(x-1)^2}{2} - \dfrac{1}{4}$
Specific behaviours
✓ separates the variables as an integration statement correctly ✓ anti-differentiates correctly using a constant ✓ determines the anti-derivative constant correctly

(c) Draw the solution curve that contains the point $(0,\ 0.5)$. (2 marks)

Solution
Shown on graph.
Specific behaviours
✓ contains the point $(0,\ 0.5)$ and the curve follows the slope field for $y > 0.5$ ✓ indicates symmetry about $y = 0$ AND indicates the curve is vertical at $y = 0$

Question 11 **(9 marks)**

Two complex numbers z_1 and z_2 are shown in the Argand plane below.



(a) Write the expression for z_1 in exact polar form. (2 marks)

Solution
$z_1 = 2cis\left(\frac{5\pi}{6}\right)$
Specific behaviours
✓ states the correct modulus ✓ states the correct argument

(b) Write the expression for z_1 in exact Cartesian form. (1 mark)

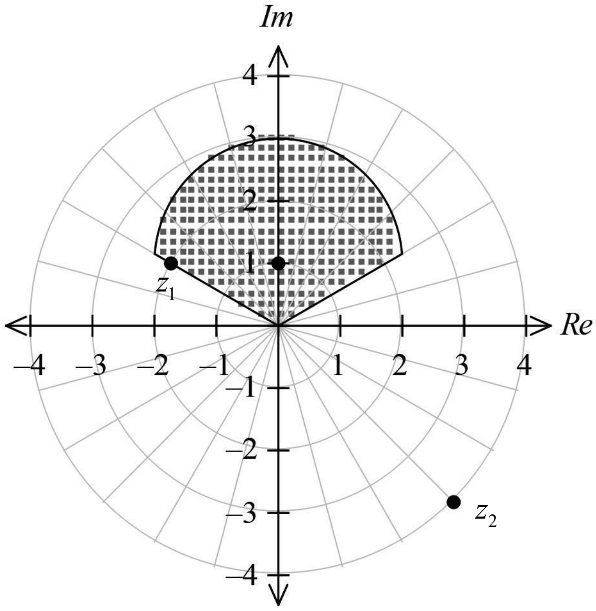
Solution
$z_1 = 2\cos\frac{5\pi}{6} + 2i\sin\frac{5\pi}{6} = 2\left(-\frac{\sqrt{3}}{2}\right) + 2i\left(\frac{1}{2}\right) = -\sqrt{3} + i$
Specific behaviours
✓ states the correct Cartesian form

Question 11 (continued)

(c) Plot the complex number iz_2 on the Argand diagram above. (2 marks)

Solution
$iz_2 = cis\left(\frac{\pi}{2}\right).4cis(a) = 4cis\left(\frac{\pi}{2}+a\right)$ i.e. multiplying by i is to rotate z_2 90° anti-clockwise about the origin
Specific behaviours
✓ determines the correct value for iz_2 ✓ plots the correct position for iz_2

(d) A sketch of the locus of a complex number z is shown below. The upper boundary of the locus is part of a circle, centred at $z = i$. Write equations or inequalities in terms of z (without using $x = \text{Re}(z)$ or $y = \text{Im}(z)$) for the indicated locus. (4 marks)



Solution
The locus is part of the interior of the circle with centre $z = i$ and radius 2. $\|z-i\| \le 2$ with $\frac{\pi}{6} \le Arg(z) \le \frac{5\pi}{6}$
Specific behaviours
✓ uses the form $\|z-c\| \le r$ ✓ states $c = i$ and $r = 2$ ✓ uses the form $\theta_1 \le Arg(z) \le \theta_2$ ✓ states $\theta_1 = \frac{\pi}{6}$, $\theta_2 = \frac{5\pi}{6}$

Question 12 **(6 marks)**

The horizontal displacement of a Ferris wheel cabin exhibits simple harmonic motion. The maximum horizontal speed is $\frac{\pi}{2}$ metres per second and its period of motion is exactly 60 seconds.

Let $x(t) = A\cos(nt)$ be the horizontal displacement after t seconds.

(a) Determine the values of A and n. (3 marks)

Solution
Period $T = 60 = \frac{2\pi}{n} \quad \therefore \quad n = \frac{\pi}{30} \quad (0.1047....)$ $v(t) = -A\left(\frac{\pi}{30}\right)\sin\left(\frac{\pi t}{30}\right) \qquad \therefore \quad Max\ v = A\left(\frac{\pi}{30}\right) = \frac{\pi}{2}$ $\therefore \quad A = 15$ Hence $x(t) = 15\cos\left(\frac{\pi t}{30}\right)$
Specific behaviours
✓ determines n correctly   ✓ differentiates and forms the correct expression for the maximum speed   ✓ determines A correctly

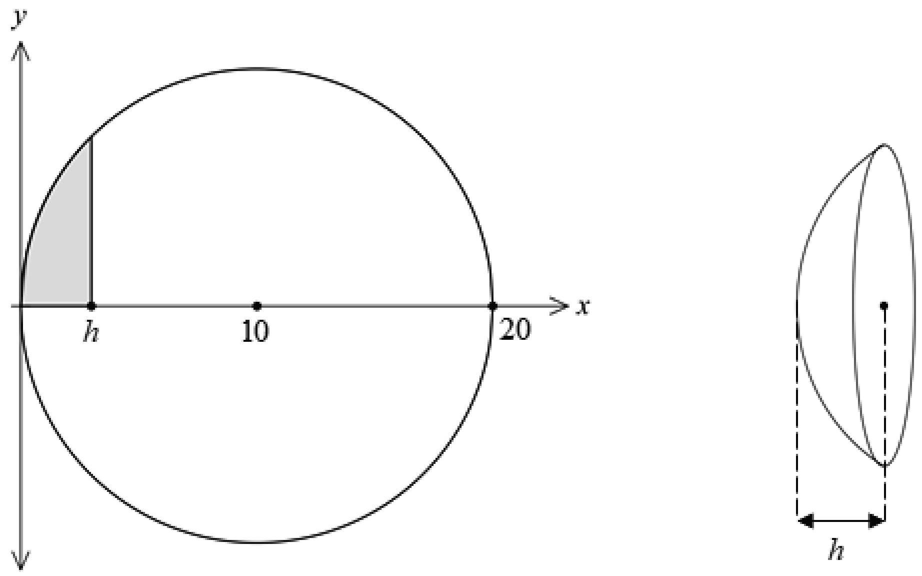
(b) Determine the horizontal acceleration, correct to the nearest 0.001 m/s^2, when the horizontal displacement is 10 metres. (3 marks)

Solution
Condition for S.H.M. is $a = -\left(\frac{\pi}{30}\right)^2 x$ $\therefore$ When $x = 10 \quad a = -\left(\frac{\pi^2}{900}\right)(10) = -0.10966\ ...$ metres/sec^2 *i.e.* acceleration is −0.110 m/sec^2 (3 d.p.)
Specific behaviours
✓ applies the condition for S.H.M. correctly   ✓ substitutes $x = 10$ correctly   ✓ determines the acceleration correct to 0.001 m/sec^2

Alternative Solution
$\therefore$ When $x = 10$ i.e. $10 = 15\cos\left(\frac{\pi t}{30}\right) \quad \therefore \quad t = 8.0316...\ \text{sec}$ $\therefore\ a(t) = -\left(\frac{\pi^2}{900}\right)(15)\cos\left(\frac{\pi t}{30}\right) \qquad a(8.0316...) = -0.110$ m/sec^2 (3 d.p.)
Specific behaviours
✓ solves for t when $x = 10$ correctly   ✓ determines the expression for the acceleration correctly   ✓ determines the acceleration correct to 0.001 m/sec^2

Question 13 **(5 marks)**

A solid spherical cap with depth h is part of a solid sphere with radius 10 cm. This cap can be generated by revolving the shaded region about the x axis.



(a) Show that the equation for the circle shown above is $x^2 + y^2 = 20x$. (1 mark)

Solution
Centre is $(10,0)$ with radius 10 $\therefore$ $(x-10)^2 + (y-0)^2 = 10^2$ i.e. $x^2 - 20x + 10^2 + y^2 = 10^2$ i.e. $x^2 + y^2 = 20x$
Specific behaviours
✓ forms the equation of the circle in centre-radius form

(b) Develop an expression for the volume of the spherical cap in terms of h. (4 marks)

Solution
The circle is given by $x^2 + y^2 = 20x$. Hence $y^2 = 20x - x^2$ Volume $V = \int_0^h \pi y^2 dx = \int_0^h \pi\left(20x - x^2\right) dx = \pi\left[10x^2 - \frac{x^3}{3}\right]_0^h$ $= \pi\left[10h^2 - \frac{h^3}{3}\right]$ . . . (1) $= \pi h^2\left(10 - \frac{h}{3}\right)$ . . . (2)
Specific behaviours
✓ writes a definite integral with the correct limits and uses correct notation ✓ writes the integrand correctly ✓ anti-differentiates correctly ✓ simplifies the volume expression in terms of h, (equivalent to 1 or 2)

Question 14 **(10 marks)**

On a Saturday afternoon, three separate family groups visit their local cinema to watch a feature movie. The cinema names this as DollarDay where the ticket prices for adults, children and pensioners are charged in whole dollar amounts.

The table below indicates the number of people in each category and the total paid for each family group.

Group	Adults	Children	Pensioners	Total cost
1	2	4	–	$108
2	3	6	–	$162
3	2	5	2	$152

Let a = the price for each adult ($)
c = the price for each child ($)
p = the price for each pensioner ($)

(a) Formulate the equations that can be used to determine the ticket prices. (1 mark)

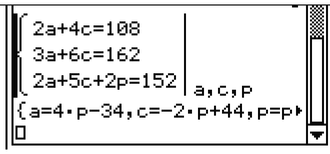
Solution
$2a+4c = 108 \quad \ldots (1)$ $3a+6c = 162 \quad \ldots (2)$ $2a+5c+2p = 152 \quad \ldots (3)$
Specific behaviours
✓ formulates the equations correctly using variables a, c, p

(b) Using the equations formed, determine the total cost for a group consisting of 1 child accompanied by 2 pensioners. (2 marks)

Solution
Consider equation $(3)-(1)$: $(2a+5c+2p)-(2a+4c) = 152-108$ i.e. $c+2p = 44$ Hence one child and 2 pensioners would cost $44.
Specific behaviours
✓ uses equations (1) and (3) to determine $c+2p$ ✓ states that the total cost is $44

Question 14 (continued)

(c) Solve simultaneously the equations formulated in part (a). (2 marks)

Solution
From CAS:    Given the pension price is p: $a=4p-34$ and $c=44-2p$ Hence there is NO unique solution (many solutions are possible for the ticket prices).
Specific behaviours
✓ obtains the correct relationships between variables ✓ states that there are MANY (non-unique) solutions

(d) Explain the geometric interpretation of the equations and the simultaneous solution. (2 marks)

Solution
Equations (1) and (2) represent that SAME plane in space. The intersection between the planes (1) and (3) gives a LINE in space, which consists of many points, which is why there is NOT a unique solution.
Specific behaviours
✓ states that the intersection gives a LINE in space ✓ states that two of the planes were identical

Now assume that the price for an adult is greater than the price for a child and that the price for a pensioner is the lowest priced ticket.

(e) Determine the ticket prices for adults, children and pensioners on DollarDay. (3 marks)

Solution
To give sensible values we require $a>0$ and $c>0$ i.e. $2p-34>0$ and $44-2p>0$ i.e. $p>8.25$ and $p<22$ Given that p is an integer (whole dollars) then $9 \le p \le 21$ Tabulating the 13 possibilities (ordered triples):

a	c	p
2	26	9
6	24	10
10	22	11
14	20	12
18	18	13
22	16	14
26	14	15
30	12	16
34	10	17
38	8	18
42	6	19
46	4	20
50	2	21

Given that $a > c > p$ there is only ONE possibility:

$a = 22$
$c = 16$
$p = 14$

Hence the prices are $22 for adults, $16 for children and $14 for pensioners.

Specific behaviours

- ✓ determines that $9 \le p \le 21$ where p is an integer (justifies a limitation on one of the variables)
- ✓ tabulates at least 2 possible ordered triples for a, c, p correctly
- ✓ states the correct triple of ticket prices

Alternative Solution

From $a + 2c = 54$ we realise that a must be EVEN.
From $c + 2p = 44$ we see that c must be EVEN.

Since $a > c$ then $3c < 54$ i.e. $c < 18$ and is EVEN
i.e. $c \le 16$

Tabulating some possibilities (ordered triples):

a	c	p
22	16	14
26	14	15
30	12	16

Given that $a > c > p$ there is only ONE possibility:

$a = 22$
$c = 16$
$p = 14$

Hence the prices are $22 for adults, $16 for children and $14 for pensioners.

Specific behaviours

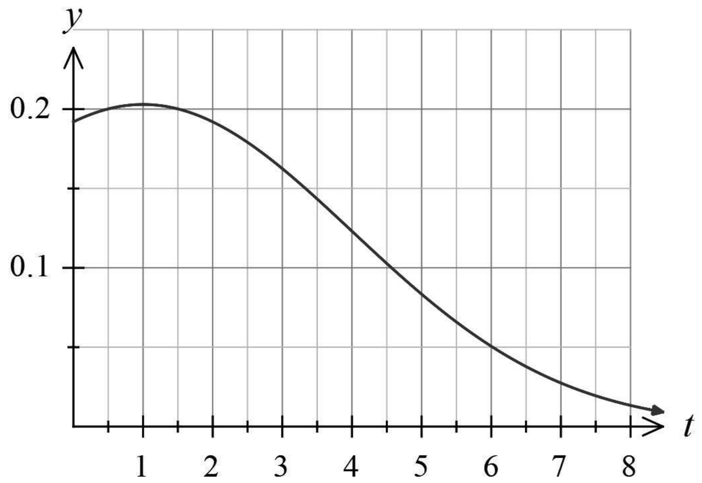
- ✓ determines that $c \le 16$ where c is an even integer (justifies a limitation on one of the variables)
- ✓ tabulates at least 2 possible ordered triples for a, c, p correctly
- ✓ states the correct triple of ticket prices

Question 15 **(9 marks)**

An experiment was conducted to measure how quickly adults respond to the request: ‘send me a text message’.

Let T = the number of hours taken for an adult to respond and send a text message.

It was found that the distribution of the population of response times for adults was given by the probability density function shown below, with mean $\mu = 3$ hours and standard deviation $\sigma = 2.4$ hours.

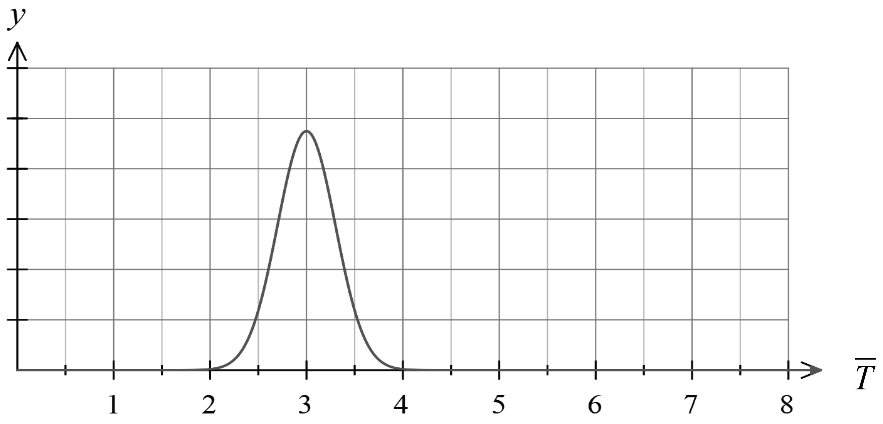


Random samples of size 64 are drawn repeatedly from the population of response times and the sample mean response time $\overline{T}$ is determined for each sample.

(a) Calculate, correct to 0.001, the probability that a sample mean response time will be between 150 minutes and 210 minutes. (3 marks)

Solution
Since $n = 64$ (significantly large), then $\overline{T} \sim N\left(3,\ \sigma_{\overline{T}}^{\ 2}\right)$ where $\sigma_{\overline{T}} = \dfrac{2.4}{\sqrt{64}} = 0.3$ Hence $P\left(2.5 < \overline{T} < 3.5\right) = 0.904$ (3 d.p.)
Specific behaviours
✓ indicates or states that the sample mean is normally distributed ✓ states the parameters of the sample mean distribution ✓ determines the probability correctly to 0.001

(b) Sketch the likely distribution of the sample mean $\overline{T}$ (for samples of size 64) on the axes below. (2 marks)



Solution
Shown above.
Specific behaviours
✓ indicates symmetry about $\overline{T}=3$ hrs ✓ indicates a standard deviation approx 0.3 hrs (very low density values for $\overline{T}>4$ and $\overline{T}<2$)

Anika, a teacher at the TekNoCrat School, theorises that as teenagers tend to check their text messages more frequently than adults, then the population mean response time for teenagers will be much lower than the population mean adult response time $\mu = 3$.

Anika is then presented with the sample mean response time for a sample gathered from an unknown source.

Sample size	Sample mean (hours)	Sample standard deviation (hours)
100	2.1	2.7

Calculations are performed and Anika concludes by stating: ‘this sample was clearly not taken from the population of adult response times. It is highly likely that this sample was taken from a sample of 100 teenagers’.

(c) Perform the necessary calculations and comment on Anika’s claim. (4 marks)

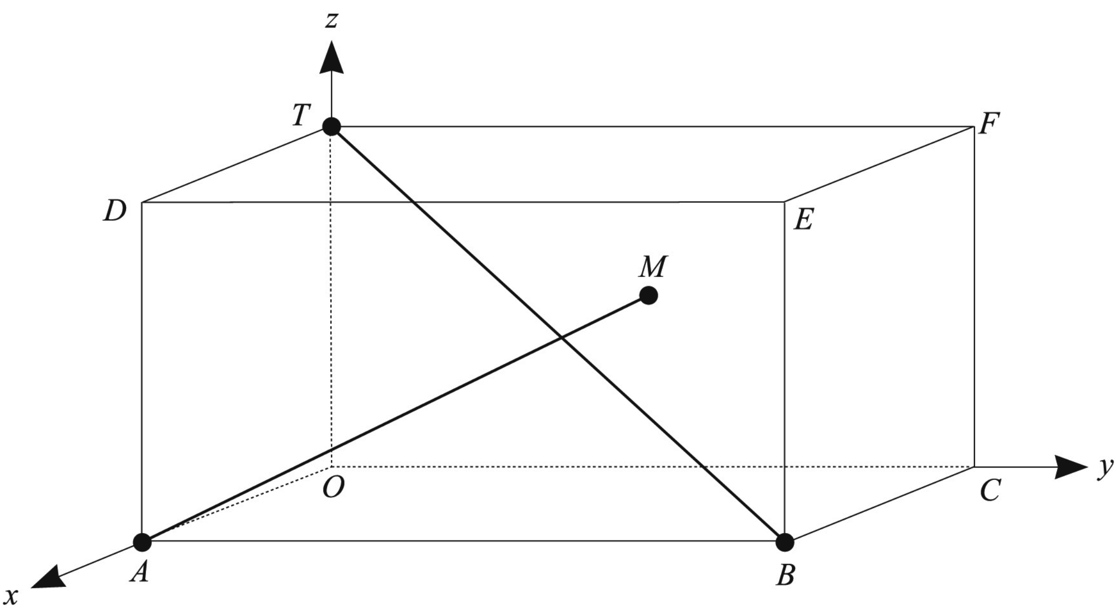
Solution
Let $\mu_T =$ the population mean for the response times of the unknown source (hrs) For the sample taken $\overline{T}=2.1$ hrs and $s=2.7$ hrs. The distribution for $\overline{T} \sim N\left(2.1,\ \sigma_{\overline{T}}^2\right)$ where $\sigma_{\overline{T}} = \dfrac{2.7}{\sqrt{100}} = 0.27$ 95% Confidence Interval for μ_T : $2.1-1.96(0.27) < \mu_T < 2.1+1.96(0.27)$ i.e. $1.5708 < \mu_T < 2.6292$ 99% Confidence Interval for μ_T :

Question 15 (continued)

$2.1-2.576(0.27) < \mu_T < 2.1+2.576(0.27)$ $1.40448 < \mu_T < 2.79552$ The ADULT population mean $\mu=3$ is NOT WITHIN either confidence interval using $\overline{T}=2.1$ and $s=2.7$. Hence the population from which the sample was drawn has a mean LOWER than the population mean for adult response times. Hence the unknown source MAY have been drawn from a different population BUT we do NOT know it was drawn from a group of teenagers. Hence Anika's claim CANNOT be accepted.
Specific behaviours
✓ determines the expected variation using $n=100$ ✓ determines an appropriate confidence interval for μ of the unknown source ✓ states that the confidence interval does NOT include the value $\mu=3$ ✓ justifies correctly that Anika's claim cannot be accepted (we do not know the unknown source was a group of teenagers)

Question 16 **(8 marks)**

A rectangular prism is defined using the coordinate system shown with $A\,(2,\,0,\,0)$, $C\,(0,\,4,\,0)$ and $T\,(0,\,0,\,3)$. Point M is the centre of the planar face $OCFT$ with coordinates $(0,\,2,\,1.5)$.



(a) Determine the vector equation for the prism's main diagonal $\overleftrightarrow{BT}$. (2 marks)

Solution
Direction vector for $\overrightarrow{BT}$ $\quad \underset{\sim}{d} = \begin{pmatrix} -2 \\ -4 \\ 3 \end{pmatrix}$
Equation $\overleftrightarrow{BT}$ $\quad \underset{\sim}{r} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2-2\lambda \\ 4-4\lambda \\ 3\lambda \end{pmatrix}$
Specific behaviours
✓ determines the direction vector for $\overrightarrow{BT}$ correctly ✓ forms the vector equation correctly using a parameter

Question 16 (continued)

(b) Determine the Cartesian equation of the sphere that contains all vertices of the rectangular prism. (3 marks)

Solution
Sphere centre S will be the midpoint of the main diagonal $\overline{BT}$ i.e. S is $\begin{pmatrix} 1 \\ 2 \\ 1.5 \end{pmatrix}$
$r^2 = CS^2 = (2-1)^2 + (4-2)^2 + (0-1.5)^2 = 7.25 \quad \therefore \; r = \frac{\sqrt{29}}{2} = 2.6925...$
Cartesian equation sphere $(x-1)^2 + (y-2)^2 + (z-1.5)^2 = 7.25$
Specific behaviours
✓ determines the centre of the sphere correctly ✓ determines the radius of the sphere correctly ✓ forms the Cartesian equation for the sphere correctly

(c) Prove, using a vector method, that line $\overleftrightarrow{AM}$ does **not** intersect $\overleftrightarrow{BT}$. (3 marks)

Solution
Equation $\overleftrightarrow{AM}$ $\quad \underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 2 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 2-2\mu \\ 2\mu \\ 1.5\mu \end{pmatrix}$
Lines will intersect if there exists real values for λ, μ where :
$\begin{pmatrix} 2-2\lambda \\ 4-4\lambda \\ 3\lambda \end{pmatrix} = \begin{pmatrix} 2-2\mu \\ 2\mu \\ 1.5\mu \end{pmatrix}$
i.e. $2-2\lambda = 2-2\mu \quad ... (1)$ From (1): $\lambda = \mu$ $4-4\lambda = 2\mu \quad ... (2)$ $\quad (2)$: $4-4\lambda = 2\lambda \quad \therefore \; \lambda = \frac{2}{3} = \mu$ $3\lambda = 1.5\mu \quad ... (3)$ $\quad (3)$: $2\lambda = \mu \quad BUT \quad \mu \neq 2\lambda$
$\therefore$ There are no values λ, μ that give an intersection. Hence line $\overleftrightarrow{AM}$ does NOT intersect $\overleftrightarrow{BT}$.
Specific behaviours
✓ states the condition for the intersection of lines in terms of the parameters ✓ forms the 3 equations comparing components for an intersection ✓ shows that no such parameter values for λ, μ exist

(c) Prove, using a vector method, that line $\overleftrightarrow{AM}$ does **not** intersect $\overleftrightarrow{BT}$. (3 marks)

Alternative Solution
Lines will NOT intersect if the distance of closest approach is greater than ZERO. Separation vector $\underset{\sim}{v} = \begin{pmatrix} -2\lambda+2\mu \\ -4\lambda-2\mu+4 \\ 3\lambda-1.5\mu \end{pmatrix}$ Closest approach is when the separation vector is PERPENDICULAR to the direction vector of each line. $\therefore\ \underset{\sim}{v}\bullet\underset{\sim}{d}_1=0 \quad 29\lambda-0.5\mu-16 = 0 \quad \ldots(1)$ $\therefore\ \underset{\sim}{v}\bullet\underset{\sim}{d}_2=0 \quad 0.5\lambda-10.25\mu+8 = 0 \quad \ldots(2)$ Solving for λ,μ: $\lambda = \frac{56}{99} = 0.5656....$ $\mu = \frac{80}{99} = 0.8080...$ $\lvert\underset{\sim}{v}\rvert = \sqrt{\left(\frac{16}{33}\right)^2+\left(\frac{4}{33}\right)^2+\left(\frac{16}{33}\right)^2} = \frac{4}{\sqrt{33}} = 0.6963\ \ldots$ Hence as $\lvert\underset{\sim}{v}\rvert > 0$ then line $\overleftrightarrow{AM}$ does NOT intersect $\overleftrightarrow{BT}$.
Specific behaviours
✓ states the condition for the closest approach ✓ forms the 2 equations for the closest approach ✓ calculates correctly the closest approach for the 2 lines

Question 17 **(12 marks)**

A researcher is interested in estimating the population mean μ (dollars) that Perth residents had spent via online shopping in December 2020. A random sample of size n gave a sample mean of \$400, a sample standard deviation s and a 95% confidence interval of width \$200.

(a) State the 95% confidence interval obtained. (1 mark)

Solution
95% CI: $400-\frac{200}{2} \leq \mu \leq 400+\frac{200}{2}$ i.e. $300 \leq \mu \leq 500$
Specific behaviours
✓ states the upper and lower limits of the interval correctly

(b) Calculate the standard deviation of the sample mean, correct to \$0.01. (2 marks)

Solution
Margin of error $= 100$ i.e. $100 = 1.96 \times \sigma\left(\overline{X}\right)$ $\therefore$ $\sigma\left(\overline{X}\right) = 51.02$
Specific behaviours
✓ forms the equation relating the margin of error and the standard deviation correctly ✓ determines the standard deviation correctly to 0.01

(c) In terms of n, what sample size would yield a 95% confidence interval of width \$50? Show your reasoning. (2 marks)

Solution
The interval width is reduced by a factor of 4, so the sample size needs to increase by a factor of $4^2 = 16$ i.e. a sample size of $16n$ is required. OR As $100 = 1.96 \times \frac{s}{\sqrt{n}}$ i.e. $s = 51.02\sqrt{n}$ Then $\frac{100}{4} = 1.96 \times \frac{s}{4\sqrt{n}}$ i.e. $25 = 1.96 \times \frac{s}{\sqrt{16n}}$ Hence NEW sample size is $16n$.
Specific behaviours
✓ uses an interval width equal to one-quarter the original ✓ states the new sample size in terms of n

Question 17 (continued)

(d) What is the probability that another sample of size $2n$ would produce a sample mean that differs from μ by more than $50? (3 marks)

Solution
$\overline{X} \sim N\left(\mu, \dfrac{s^2}{2n}\right)$ i.e. $\overline{X} \sim N\left(\mu, \dfrac{51.02^2}{2}\right)$ $\therefore \sigma\left(\overline{X}\right) = 36.0768...$ Require $P\left(\left\lvert\overline{X} - \mu\right\rvert > 50\right) = P\left(\lvert z\rvert > \dfrac{50}{36.0768}\right) = P\left(\lvert z\rvert > 1.3859\right)$ $= 2(0.083)$ $= 0.166$
Specific behaviours
✓ determines the standard deviation for the sample size $2n$ correctly   ✓ forms the correct probability statement   ✓ calculates the correct probability

Four different confidence intervals (A, B, C and D) are obtained for the mean amount spent via online shopping by Perth residents in December 2020.

Confidence interval	Sample size	Sample standard deviation	Confidence level
A	n	s	95%
B	n	s	99%
C	$2n$	s	95%
D	n	$0.8s$	95%

(e) Which of the confidence intervals (A, B, C or D) contains μ, the population mean expenditure for online shopping in December 2020? Justify your answer. (2 marks)

Solution
Since the true value of μ is unknown, we CANNOT determine which interval contains the true mean. This is due to the inherent nature of random sampling.
Specific behaviours
✓ states we cannot determine which interval contains μ ✓ states that either μ is unknown OR refers to the nature of random sampling

Question 17 (continued)

(f) For each of the following, state the confidence interval that has the smaller width. Justify your answers.

(i) A and B. (1 mark)

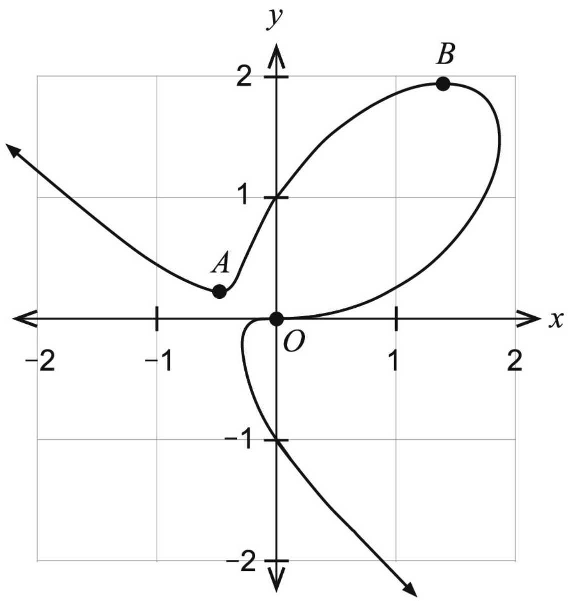
Solution
Confidence interval A will have the smaller width since the level of confidence 95% is less than that of B 99%.
Specific behaviours
✓ justifies why A will have the smaller width

(ii) C and D. (1 mark)

Solution
Need to compare the standard deviation of the sample means: $C: \quad \sigma(\overline{X}) = \frac{s}{\sqrt{2n}} = 0.707\left(\frac{s}{\sqrt{n}}\right)$ $D: \quad \sigma(\overline{X}) = \frac{0.8s}{\sqrt{n}} = 0.8\left(\frac{s}{\sqrt{n}}\right)$ Hence confidence interval C will have the smaller width.
Specific behaviours
✓ justifies why C will have the smaller width by correctly comparing the respective standard deviations of the sample mean

Question 18 **(6 marks)**

The equation $x^3 + y^3 = 3xy + y$ implicitly defines the curve shown below.



(a) Using implicit differentiation obtain the expression for $\dfrac{dy}{dx}$. (3 marks)

Solution
$\dfrac{d}{dx}\left(x^3+y^3\right) = \dfrac{d}{dx}(3xy+y)$ $\therefore\ 3x^2 + 3y^2\left(\dfrac{dy}{dx}\right) = 3x\left(\dfrac{dy}{dx}\right) + 3y + \left(\dfrac{dy}{dx}\right)$ ..... (1) i.e. $\dfrac{dy}{dx} = \dfrac{3\left(x^2-y\right)}{1+3x-3y^2} = \dfrac{3\left(y-x^2\right)}{3y^2-3x-1}$
Specific behaviours
✓ obtains the left hand expression of statement (1) ✓ obtains the right hand expression of statement (1) ✓ obtains the correct expression for $\dfrac{dy}{dx}$ (or its equivalent)

Question 18 (continued)

The slope of the curve at the origin O and points A and B is equal to zero.

(b) Show that the equation that determines the x coordinates for points A and B is given by $x^4 - 2x - 1 = 0$ and hence determine the coordinates for point A correct to 0.001.

(3 marks)

Solution
For a slope of zero we require $\frac{3(x^2 - y)}{1 + 3x - 3y^2} = 0 \quad i.e. \quad y = x^2$ (numerator zero) Hence substituting $y = x^2$ into the equation $x^3 + y^3 = 3xy + y$ obtains $x^3 + (x^2)^3 = 3x(x^2) + x^2$ i.e. $x^3 + x^6 = 3x^3 + x^2$ i.e. $x^6 - 2x^3 - x^2 = 0$ $\therefore \; x^2(x^4 - 2x - 1) = 0 \quad \ldots (2)$ Since for points A, B $x \neq 0$ then it must be that $x^4 - 2x - 1 = 0$. Using CAS we obtain point A as $(-0.475,\ 0.225)$ correct to 0.001 Note: point B is $(1.395,\ 1.947)$
Specific behaviours
✓ deduces that $y = x^2$ is required for a zero slope ✓ obtains equation (2) that determines A, B ✓ obtains the coordinates for point A correct to 0.001

Question 19 **(14 marks)**

Using the correct technique, Olympic ski jumpers can slow down their descent, by creating lift to counteract gravity. These jumpers must land successfully to have their distance recorded and land on sloped ground to prevent serious injury.

For copyright reasons this text cannot be reproduced in the online version of this document.

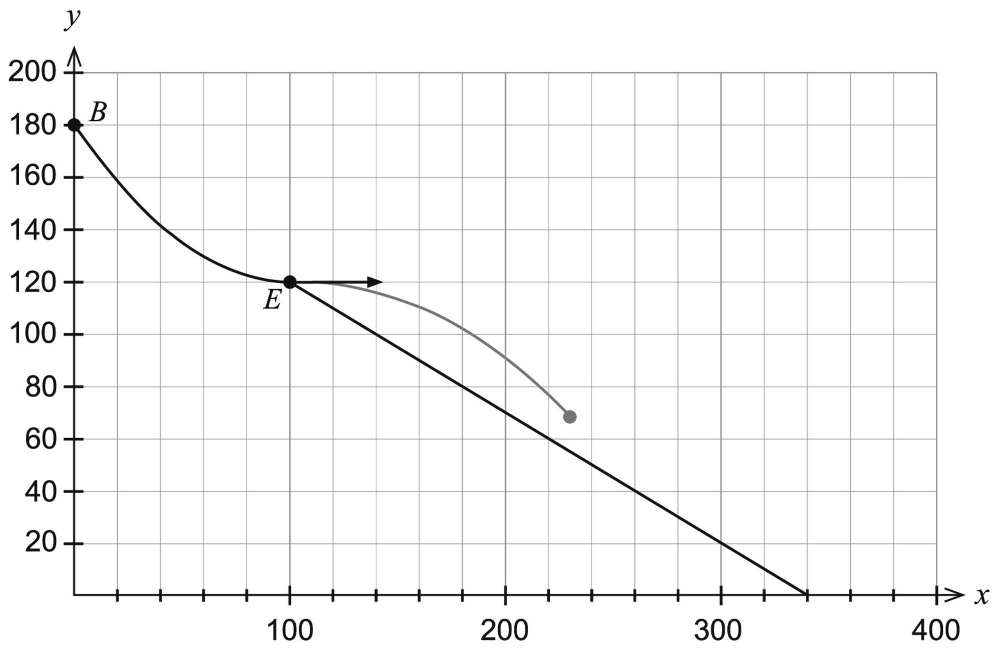
A skier begins his descent at point B accelerating down the ramp. At the end of the ramp the skier is travelling horizontally at point E at 32 metres per second (115.2 kilometres per hour).

Let t = the number of seconds in flight after point $E\ (100,120)$.

$h(t)$ = the height of the skier above the horizontal ground $y=0$ (metres)

$x(t)$ = the horizontal position of the skier (metres)

The sloped ground for landing is given by $y = 170 - 0.5x$ where $100 \le x \le 340$.



The ski jumper's suit and skis decrease the horizontal velocity $x'(t)$ so that $x'(t) = 32e^{-0.05t}$.

(a) Show that $x(t) = 740 - 640e^{-0.05t}$. (2 marks)

Solution
$x(t) = \int 32e^{-0.05t}dt = \dfrac{32}{-0.05}e^{-0.05t} + c$ i.e. $x(t) = -640e^{-0.05t} + c$ Using $x(0)=100$ then $100 = -640(1) + c$ $\therefore\ c = 740$
Specific behaviours
✓ anti-differentiates $x'(t)$ correctly using a constant   ✓ uses $x(0)=100$ correctly to determine the constant

Question 19 (continued)

It is found that the expression for the position vector for the skier during the flight is given by:

$$\underset{\sim}{r}(t) = \begin{pmatrix} 740 - 640e^{-0.05t} \\ 120 - 2.5t^2 \end{pmatrix}$$

(b) Calculate the height of the skier above the sloped ground after 3 seconds of flight, correct to the nearest 0.01 metre. (3 marks)

Solution
At $t = 3$ $\quad r(3) = \begin{pmatrix} 189.1468... \\ 97.5 \end{pmatrix}$ For $x = 189.1468..$ Sloped ground $y = 170 - 0.5(189.1468..) = 75.42655...$ Hence skier height ABOVE the ground $= 97.5 - 75.42655... = 22.073...$ m i.e. The skier is 22.07 metres above the sloped ground after 3 seconds.
Specific behaviours
✓ calculates both components for $\underset{\sim}{r}(3)$ correctly ✓ calculates the sloped ground height correctly ✓ determines the height of the skier above the ground correct to 0.01 metres

(c) Determine the vertical lift s (m/s^2) provided by the skier's suit and equipment in the descent if $\dfrac{d^2h}{dt^2} = s - 9.8$, where s is a constant. (3 marks)

Solution
Using $\dfrac{d^2h}{dt^2} = s - 9.8$ then $h'(t) = (s-9.8)t + c$ Since $h'(0) = 0$ then $c = 0$ i.e. $h'(t) = (s-9.8)t$ $\therefore\ h(t) = \dfrac{(s-9.8)}{2}t^2 + k$ $\quad$ Since $h(0) = 120$ then $k = 120$ i.e. $h(t) = \dfrac{(s-9.8)}{2}t^2 + 120$ Hence as $h(t) = 120 - 2.5t^2$ then $\dfrac{s-9.8}{2} = -2.5$ Solving gives $s = 4.8$ ms^{-2}.
Specific behaviours
✓ performs appropriate calculus correctly ✓ forms an equation to determine the value of s ✓ determines the value of s correctly

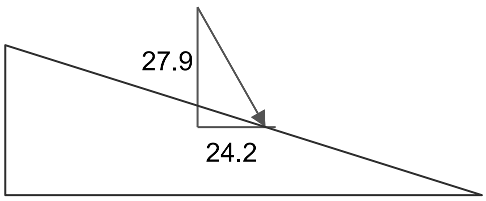
It can be shown that the Cartesian equation for the skier's flight is given by:

$$y = 120 - 1000\left(\ln\left(\frac{740-x}{640}\right)\right)^2$$

(d) Calculate the time taken for the skier to land on the sloped ground, correct to the nearest 0.01 second. (3 marks)

Solution
We need to determine the landing point on the sloped ground. Solving simultaneously: $y = 170 - 0.5x$ $y = 120 - 1000\left(\ln\left(\frac{740-x}{640}\right)\right)^2$ From CAS: $(100,120)$ and $(255.915887,\ 42.04205652)$ $\therefore\ 42.04205652 = 120 - 2.5t^2$ OR $255.915887 = 740 - 640e^{-0.05t}$ From CAS: $t = 5.584189949...$ sec Hence the skier will take 5.58 seconds to land on the sloped ground.
Specific behaviours
✓ solves simultaneously to correctly determine the landing position ✓ forms the equation to solve for t correctly ✓ solves for t correct to 0.01 seconds

(e) Calculate the angle at which the skier impacts the sloped ground, correct to the nearest 0.1 degree. (3 marks)

Solution
We need to determine the velocity vector $r'(5.5841)$. $r'(t) = \begin{pmatrix} 32e^{-0.05t} \\ -5t \end{pmatrix}$ $\therefore\ \underset{\sim}{r}'(5.58418...) = \begin{pmatrix} 24.2042... \\ -27.9209.. \end{pmatrix}$  Angle of landing $= \tan^{-1}\left(\frac{27.9209}{24.2042}\right) - \tan^{-1}(0.5)$ $= 49.0784° - 26.5650°$ $= 22.513°$ Hence the skier lands at an angle of 22.5° to the sloped ground.
Specific behaviours
✓ determines the velocity vector $\underset{\sim}{r}'(5.5841)$ for the landing correctly ✓ forms the correct expression to determine the angle of landing ✓ determines the angle of landing correctly

ACKNOWLEDGEMENTS

Question 19 U.S. Ski & Snowboard. (2018). *Ben Loomis nordic combined 2018 US Olympic team trails at the UOP* [Photograph]. Retrieved April, 2021, from https://usskiandsnowboard.smugmug.com/Jumping-Nordic-Combined/201718-Ski-Jumping-Nordic-Combined/2018-Olympic-Team-Trials-Nordic-Combined/i-ZS4JBFk/A

Published by the School Curriculum and Standards Authority of Western Australia
303 Sevenoaks Street
CANNINGTON WA 6107