2023/74320

Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS SPECIALIST

Calculator-assumed

ATAR course examination 2023

Marking key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

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Section Two: Calculator-assumed **65% (89 Marks)**

Question 9 **(6 marks)**

The Cartesian equation of a sphere is given as $x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$.

(a) Write the equation of the sphere in vector form. (3 marks)

Solution
Completing the square: $(x-2)^2 + (y+1)^2 + (z-3)^3 - 2^2 - 1^2 - 3^2 + 5 = 0$ i.e. $(x-2)^2 + (y+1)^2 + (z-3)^3 = 9 \quad ..... \quad (1)$ $\therefore$ Centre is $(2,-1,3)$ Radius $= \sqrt{9} = 3$ Vector equation of sphere: $\left\| \underset{\sim}{r} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right\| = 3$
Specific behaviours
✓ completes the square correctly to obtain statement (1) ✓ determines the centre and radius correctly   ✓ forms the vector equation of the sphere correctly

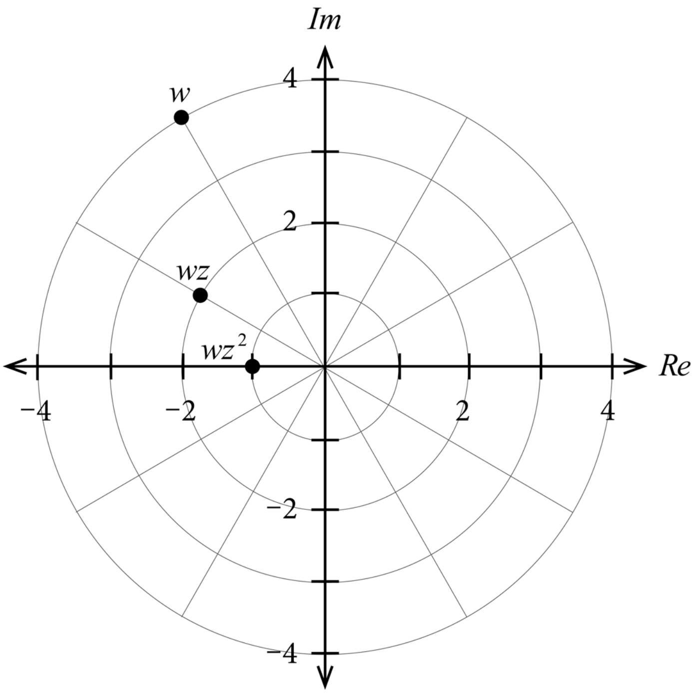
A line has vector equation $\underset{\sim}{r} = \begin{pmatrix} 7 \\ -1 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$.

(b) Determine the point(s) of intersection between the line and the sphere. (3 marks)

Solution
Substitute $\underset{\sim}{r} = \begin{pmatrix} 7+3\lambda \\ -1-\lambda \\ 9+4\lambda \end{pmatrix}$ into $\left\| \underset{\sim}{r} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right\| = 3$.   i.e. $\left\| \begin{pmatrix} 7+3\lambda \\ -1-\lambda \\ 9+4\lambda \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right\| = 3 \quad \therefore \quad \left\| \begin{pmatrix} 5+3\lambda \\ -\lambda \\ 6+4\lambda \end{pmatrix} \right\| = 3$ i.e. $(5+3\lambda)^2 + (-\lambda)^2 + (6+4\lambda)^2 = 9 \quad ... \quad (2)$ Solving using CAS: $\lambda = -1, \quad \lambda = -2$.   Hence the 2 points of intersection are $(4, 0, 5)$ and $(1, 1, 1)$.
Specific behaviours
✓ substitutes correctly to obtain statement (2) ✓ solves for the values of λ correctly   ✓ determines the points of intersection correctly (either Cartesian or vector form)

Question 10 **(7 marks)**

The complex number $w = 4cis\left(\frac{2\pi}{3}\right)$ is shown in the Argand diagram, along with the complex numbers wz and wz^2.



(a) Express wz and wz^2 in exact polar form. (2 marks)

Solution
$wz = 2cis\left(\frac{5\pi}{6}\right) \qquad wz^2 = cis(\pi)$
Specific behaviours
✓ writes the correct modulus for each complex number ✓ writes the correct argument for each complex number or ✓ writes the correct modulus and argument for one complex number ✓ writes the correct modulus and argument for the other complex number

Question 10 (continued)

Consider the geometric transformation(s) applied to transform $w \to wz \to wz^2$ etc.

(b) Describe the geometric transformation(s) performed by successive multiplication by z. (2 marks)

Solution
Successive multiplication by z results in the modulus changing by a factor of $\frac{1}{2}$ and the argument increasing by $\frac{\pi}{6}$ i.e. 30°. Geometric description: Each vector is scaled by a factor of 0.5. Rotation anti-clockwise (about origin) by 30°.
Specific behaviours
✓ describes the change in the modulus as a dilation by factor 0.5 ✓ describes the change in the argument as an anti-clockwise rotation by 30°

(c) Determine z in exact polar form. (1 mark)

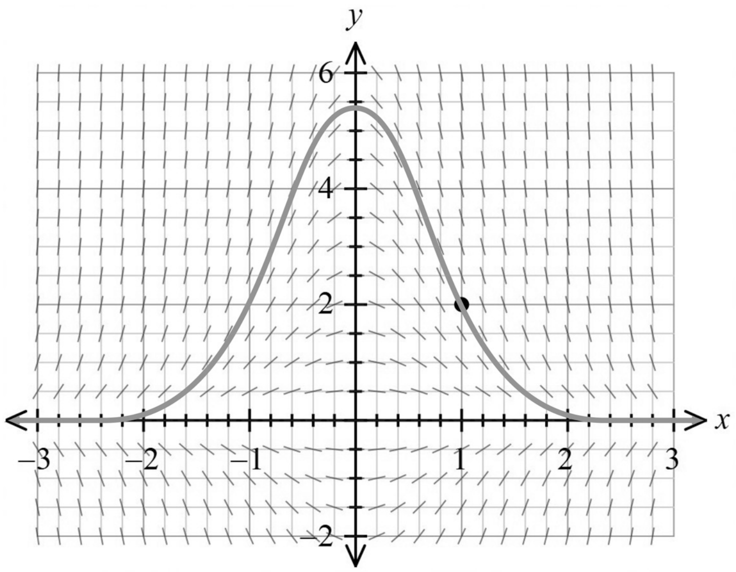
Solution
$z = \frac{1}{2} cis\left(\frac{\pi}{6}\right)$
Specific behaviours
✓ determines the correct polar form for z

(d) Describe the geometric transformation(s) performed by successive multiplication by z^{-1}. (2 marks)

Solution
$z^{-1} = \left(\frac{1}{2} cis\left(\frac{\pi}{6}\right)\right)^{-1} = 2\,cis\left(-\frac{\pi}{6}\right)$ Successive multiplication by z^{-1} results in the modulus changing by a factor of 2 and the argument decreasing by $\frac{\pi}{6}$ i.e. 30°. Geometric description: Each vector is scaled by a factor of 2. Rotation clockwise (about origin) by 30°.
Specific behaviours
✓ describes the change in the modulus as an enlargement/dilation by factor 2 ✓ describes the change in the argument as a clockwise rotation by 30°

Question 11 **(5 marks)**

A slope field is given by the equation $\frac{dy}{dx} = k(xy)$ where k is a constant.



(a) The value of the slope field at the point $(1, 2)$ is equal to -4. Determine the value of the constant k. (2 marks)

Solution
At $(1,2)$ $\frac{dy}{dx} = -4 = k(1)(2)$ $\therefore k = -2$ i.e. $\frac{dy}{dx} = -2(xy)$
Specific behaviours
✓ states that $\frac{dy}{dx} = -4$ ✓ solves correctly to determine k

(b) Determine the equation for the solution curve that contains the point $(1, 2)$ and draw this curve on the diagram above. (3 marks)

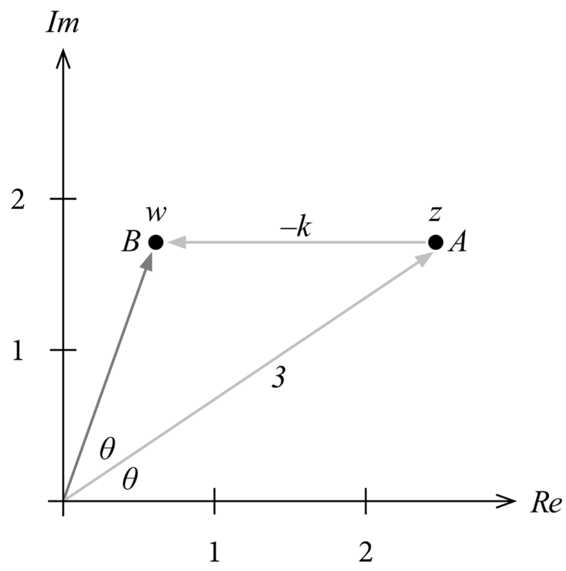
Solution
From $\frac{dy}{dx} = -2xy$ $\therefore \int \frac{dy}{y} = \int -2x\,dx$ ... (1) $\therefore \ln y = -x^2 + c$ $\therefore y = e^{-x^2+c}$ Using $(1, 2)$ $2 = e^{-1+c}$ $\therefore c = \ln 2 + 1 = 1.6931...$ $\therefore y = e^{-x^2+\ln 2+1} = 2e\left(e^{-x^2}\right)$ or $2e^{-x^2+1}$
Specific behaviours
✓ separates the variables correctly to form statement (1) ✓ determines the equation for the solution curve through $(1, 2)$ ✓ draws the solution curve with symmetry about $x = 0$ to follow the slope field

Question 12 **(6 marks)**

Complex numbers z and w are shown in the Argand diagram below. It is known that:

$|z| = 3$, $Arg(z) = \theta$ where $0 < \theta < \frac{\pi}{4}$

$w = z - k$ such that $Arg(w) = 2\theta$ where $Im(k) = 0$, $k > 0$.



(a) Represent the given information on the Argand diagram. (3 marks)

Solution
Shown on diagram.
Specific behaviours
✓ indicates $|z| = 3$ and $Arg(z) = \theta$ using line segments or vectors ✓ indicates horizontal length equal to k ✓ indicates $Arg(w) = 2\theta$

(b) Determine a simplified expression for k in terms of θ. Justify your answer. (3 marks)

Solution
Applying the Cosine rule in ΔOAB : $3^2 = k^2 + k^2 - 2(k)(k)\cos(\pi - 2\theta)$
i.e. $9 = 2k^2 - 2k^2\cos(-2\theta)$
i.e. $9 = 2k^2 + 2k^2\cos(2\theta)$
i.e. $9 = 2k^2(1+\cos(2\theta))$
$\therefore\ 9 = 2k^2(2\cos^2\theta) = 4k^2\cos^2\theta$ This yields $k = \dfrac{3}{2\cos\theta}$.
Specific behaviours
✓ states that $\lvert w \rvert = k$ or refers to $s\angle OAB = \theta = s\angle AOB$
✓ forms an equation relating k, θ using appropriate trigonometry
✓ obtains the correct simplified expression for k in terms of θ

Alternative Solution
Applying the Sine rule in ΔOAB :
$\dfrac{k}{\sin\theta} = \dfrac{3}{\sin(\pi - 2\theta)}$
i.e. $k = \dfrac{3\sin\theta}{\sin(2\theta)} = \dfrac{3\sin\theta}{2\sin\theta\cos\theta}$
This yields $k = \dfrac{3}{2\cos\theta}$.
Specific behaviours
✓ states that $\lvert w \rvert = k$ or refers to $s\angle OAB = \theta = s\angle AOB$
✓ forms an equation relating k, θ using appropriate trigonometry
✓ obtains the correct simplified expression for k in terms of θ

Alternative Solution
Let $z = 3\cos\theta + (3\sin\theta)i$
$\therefore\ w = (3\cos\theta - k) + (3\sin\theta)i$
But we also have $w = k\cos 2\theta + (k\sin 2\theta)i$
Equating imaginary parts: $k\sin 2\theta = 3\sin\theta$
$\therefore\ k = \dfrac{3\sin\theta}{\sin 2\theta}$
This yields $k = \dfrac{3}{2\cos\theta}$.
Specific behaviours
✓ states that $\lvert w \rvert = k$ or refers to $s\angle OAB = \theta = s\angle AOB$
✓ forms an equation relating k, θ using the real or imaginary parts
✓ obtains the correct simplified expression for k in terms of θ

Question 13 **(11 marks)**

A factory produces boxes of breakfast cereal with a labelled weight of 1.00 kg.

Let μ denote the population mean and σ denote the population standard deviation of the weight of the boxes. The factory sets the packaging process to a mean weight $\mu = 1.01$ kg with a standard deviation $\sigma = 0.05$ kg.

To maintain quality, a random sample of 400 boxes is taken each day and weighed. Let $\overline{X}$ denote the sample mean weight.

(a) State the distribution for $\overline{X}$ and its parameters. (3 marks)

Solution
Since $n = 400 > 30$, then $\overline{X} \sim N\left(\mu,\ \sigma_{\overline{X}}^{\ 2}\right)$ i.e. normally distributed and centred with a mean 1.01 kg and standard deviation of the sample mean $\sigma\left(\overline{X}\right) = \dfrac{0.05}{\sqrt{400}} = 0.0025$ kg i.e. $\overline{X} \sim N\left(1.01,\ 0.0025^2\right)$.
Specific behaviours
✓ states that the sample mean will be normally distributed ✓ states the mean of the distribution is 1.01 kg ✓ states the standard deviation is 0.0025 kg

(b) Determine the probability that the sample mean is more than 5 g above the labelled weight. (2 marks)

Solution
$P\left(\overline{X} > 1.005\right) = P\left(Z > \dfrac{1.005 - 1.01}{0.0025}\right) = P\left(Z > -2\right)$ $= 0.9772$
Specific behaviours
✓ states the critical weight 1.005 kg ✓ determines the correct probability

The sample mean on a particular day is $\overline{x} = 1.05$ kg, while the sample standard deviation is $s = 50$ g.

(c) Determine a 95% confidence interval, correct to 0.001 kg, for the population mean weight based on this sample. (2 marks)

Solution
95% CI: $1.05 - 1.959964 \times 0.0025 < \mu < 1.05 + 1.959964 \times 0.0025$ $\therefore\ 1.045 < \mu < 1.055$ kg
Specific behaviours
✓ uses the correct critical z score for 95% confidence ✓ determines the bounds correctly to 0.001 kg

Anja, a quality control officer, wants a 95% confidence interval based on a sample size of 100 with a width of no more than 0.1 kg.

(d) What is the maximum standard deviation for this confidence interval? (2 marks)

Solution
Half-width $w = 0.05 = 1.959964 \times \frac{s}{\sqrt{100}}$ Solving gives $s = 0.2551$ kg.
Specific behaviours
✓ uses the correct expression for the half-width ✓ determines the value of the standard deviation correctly

Over the next 50 days, Ben, who is a data collection agent, takes random samples of size 100 each day and a 95% confidence interval is calculated for each sample. Ten of these 50 intervals (20% of the intervals) have a lower bound that is less than 1.00 kg. Ben claims that this indicates that the mean weight of the packaging is set too low.

(e) Is Ben correct? Justify your response. (2 marks)

Solution
Ben is NOT correct. This is a result of random sampling, and the confidence intervals will vary according to the sample data. There is no guarantee that 95% of the confidence intervals will lie above a certain value. All that is guaranteed is that in the long run 95% of the confidence intervals are expected to contain the true population mean μ.
Specific behaviours
✓ states that Ben is not correct ✓ justifies appropriately, based on random sampling

Question 14 **(11 marks)**

Plane P_1 has Cartesian equation: $z = 2x + y + 4$.

Line L has equation given by: $\underset{\sim}{r} = \begin{pmatrix} 2-\lambda \\ 1+\lambda \\ 2\lambda \end{pmatrix}$

(a) Determine a vector that is perpendicular to plane P_1. (2 marks)

Solution
The normal vector $\underset{\sim}{n}_1$ for plane P_1 will be perpendicular to the plane. The Cartesian equation (in standard form) is $2x + y - z = -4$. Hence $\underset{\sim}{n}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.
Specific behaviours
✓ uses the standard form for the plane $2x + y - z = -4$ ✓ states the normal vector correctly

(b) Write the equation for plane P_1 in vector form. (2 marks)

Solution
Since the standard form for P_1 is $2x + y - z = -4$ then we can write that: $\underset{\sim}{r} \bullet \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = -4$
Specific behaviours
✓ uses the correct normal vector $\underset{\sim}{n}_1$ ✓ forms the vector equation for plane P_1 correctly

Alternative Solution
Points in plane P_1: x | y | z -2 | 0 | 0 0 | -4 | 0 0 | 0 | 4 Vector equation P_1: $\underset{\sim}{r} \bullet \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$ i.e. $\underset{\sim}{r} \bullet \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = -4$
Specific behaviours
✓ determines a point in plane P_1 correctly ✓ forms the vector equation for plane P_1 correctly

Alternative Solution
Direction vectors that lie in plane P_1: $\underset{\sim}{v}_1 = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}$ $\underset{\sim}{v}_2 = \begin{pmatrix} -2 \\ 0 \\ -4 \end{pmatrix}$ $\underset{\sim}{v}_3 = \begin{pmatrix} 0 \\ -4 \\ -4 \end{pmatrix}$ Vector equation P_1: $\underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + \lambda(\underset{\sim}{v}_1) + \mu(\underset{\sim}{v}_2)$ (or equivalent)
Specific behaviours
✓ determines two vectors that lie in plane P_1 ✓ forms the vector equation for plane P_1 correctly using two parameters

(c) Determine the acute angle, correct to the nearest degree, between plane P_1 and line L.
(3 marks)

Solution
This angle is determined by finding the angle between the normal to P_1 and line L.   Consider forming the dot product: $\underset{\sim}{n}_1 \bullet \underset{\sim}{d}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = -2+1-2 = -3$ $\therefore \ \lvert\underset{\sim}{n}_1\rvert\lvert\underset{\sim}{d}_1\rvert\cos\theta = -3$ $\therefore \ (\sqrt{6})(\sqrt{6})\cos\theta = -3$ $\therefore \ \cos\theta = -0.5$ i.e. $\lvert\theta\rvert = 60^o$ (solution of the smallest magnitude)   Hence the angle between the line and the plane is $90^o - \lvert\theta\rvert = 30^o$.
Specific behaviours
✓ considers the angle between the plane normal and line direction vector   ✓ determines the angle between the normal and the line correctly   ✓ deduces the acute angle between the plane and the line correctly

Question 14 (continued)

(d) Obtain the Cartesian equation of the plane P_2 that contains the line L and is perpendicular to plane P_1. (4 marks)

Solution
Let the normal vector for plane P_2 be $\underset{\sim}{n}_2$.   Since plane $P_2 \perp P_1$ then $\underset{\sim}{n}_2 \perp \underset{\sim}{n}_1$.   Also as plane P_2 that contains the line L_1 then $\underset{\sim}{n}_2 \perp \underset{\sim}{d}_1$.   Hence let $\underset{\sim}{n}_2 = \underset{\sim}{n}_1 \times \underset{\sim}{d}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$ or use $\underset{\sim}{n}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.   Any point of line L_1 can be used as the point that will be in plane P_2 i.e. $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.   Vector equation for plane P_2: $\underset{\sim}{r} \bullet \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 1$ i.e. Cartesian equation for plane P_2: $x - y + z = 1$
Specific behaviours
✓ states that the normal vector $\underset{\sim}{n}_2 = \underset{\sim}{n}_1 \times \underset{\sim}{d}_1$ ✓ determines the normal vector $\underset{\sim}{n}_2$ correctly   ✓ uses a known point on line L correctly as the point on plane P_2 ✓ determines the Cartesian equation for plane P_2 correctly

Alternative Solution
Vectors $\underset{\sim}{n}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\underset{\sim}{d} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ are vectors in plane P_2.
Hence plane P_2 is given by: $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$
$\therefore \quad x = 2 - \lambda + 2\mu \quad ...(1)$ $y = 1 + \lambda + \mu \quad ...(2)$ $z = 2\lambda - \mu \quad ...(3)$
$(2)+(3): \quad y + z = 1 + 3\lambda \quad ...(4)$ $(1)-2(2): \quad x - 2y = -3\lambda \quad ...(5)$
Hence $(4)+(5): \quad x - y + z = 1$
i.e. Cartesian equation for plane P_2: $\quad x - y + z = 1$
Specific behaviours
✓ writes the equation for plane P_2 using directions $\underset{\sim}{n}_1$ and $\underset{\sim}{d}$ and a point on line L ✓ forms the three parametric equations in terms of parameters $\lambda, \ \mu$ ✓ eliminates a parameter from a pair of equations ✓ determines the Cartesian equation for plane P_2 correctly

Question 15 **(9 marks)**

The WeLuvYas Bank extends personal loans to approved customers. A random sample of n personal loans is taken. A 99% confidence interval for the population mean loan μ (in thousands of dollars) based on this sample is $10.2 < \mu < 25.4$.

(a) What is the mean personal loan $\bar{x}$ for this sample? (2 marks)

Solution
$\bar{x} = \dfrac{10.2+25.4}{2} = 17.8$ Hence the mean personal loan was $17 800.
Specific behaviours
✓ calculates the midpoint of the confidence interval correctly ✓ states the personal loan amount in dollars

(b) Calculate the standard deviation of the sample mean. (2 marks)

Solution
Half-width $w = 17.8 - 10.2 = 7.6$ $\therefore$ $7.6 = 2.5758 \times \sigma\left(\overline{X}\right)$ i.e. $\sigma\left(\overline{X}\right) = 2.95$ (2 d.p.) i.e. standard deviation is $2950
Specific behaviours
✓ forms the expression for half-width of interval in terms of the standard deviation ✓ calculates the standard deviation correctly

Ali exclaims excitedly 'everyone here at WeLuvYas is 99% certain that the true population mean personal loan is within the interval $10.2 < \mu < 25.4$'.

(c) State **two** reasons why Ali is not correct. (2 marks)

Solution
Ali is not correct. It can be said that: 1. A single confidence interval either contains μ or it doesn't. 2. The value of μ is unknown so we do not know if any given CI contains μ. 3. If we repeatedly take samples of size n then we will find that approximately 99% of these intervals will contain the true value of μ. 4. The value of n may be less than 30, meaning that the distribution of the sample mean may not be distributed, hence the 99% confidence interval may not be valid. Note: it is insufficient to only refer to the 'inherent nature of random sampling'.
Specific behaviours
✓ states one reason ✓ states a second reason

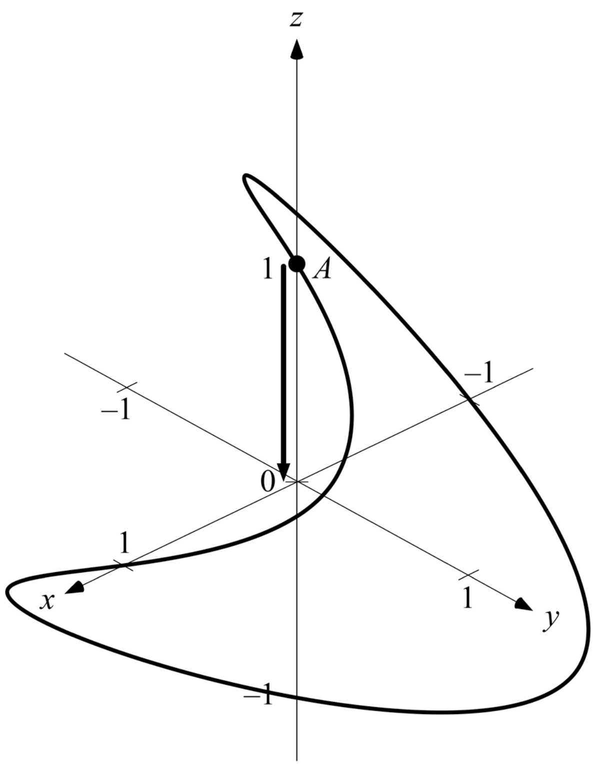
A data analyst discovers that the sample size was actually $2n$. In addition to this, the sample mean was actually $2000 more than that originally determined.

(d) Re-calculate the 99% confidence interval for the population mean on the basis of the updated information. (3 marks)

Solution
The confidence interval changes is two ways. 1. The whole interval is translated upwards by 2. 2. The standard error $\sigma\left(\overline{X}\right)$ and consequently the width of the interval is scaled by a factor of $\frac{1}{\sqrt{2}}$: $w = 2.5758\times\frac{s}{\sqrt{2n}} = \left(2.5758\times\frac{s}{\sqrt{n}}\right)\times\frac{1}{\sqrt{2}}$ $= (7.6)\times\frac{1}{\sqrt{2}} = 5.37$ New confidence interval : $17.8+2-5.37 < \mu < 17.8+2+5.37$ i.e. $14.43 < \mu < 25.17$
Specific behaviours
✓ indicates the midpoint of the confidence interval is increased by 2 ✓ calculates the new standard deviation correctly ✓ calculates the new confidence interval correctly

Question 16 **(5 marks)**

A fly moves around a path given by a three dimensional curve. The fly's path begins at point A and is shown below. Its position vector is specified by $\underset{\sim}{r}(t) = \begin{pmatrix} \sin t \\ \sin 2t \\ \cos t \end{pmatrix}$ metres where $0 \le t \le 2\pi$ seconds.



(a) Determine the initial acceleration vector and indicate this clearly on the diagram above. (3 marks)

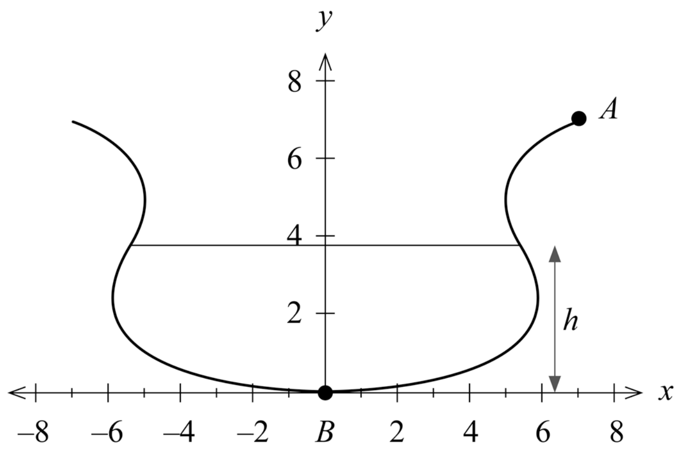
Solution
$\underset{\sim}{v}(t) = \underset{\sim}{r}'(t) = \begin{pmatrix} \cos t \\ 2\cos 2t \\ -\sin t \end{pmatrix} \qquad \underset{\sim}{a}(t) = \underset{\sim}{r}''(t) = \begin{pmatrix} -\sin t \\ -4\sin 2t \\ -\cos t \end{pmatrix}$
$\therefore \ \underset{\sim}{a}(0) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ i.e. the acceleration vector is acting directly downwards with a magnitude 1 m/sec^2 (shown on diagram).
Specific behaviours
✓ determines the acceleration vector function correctly ✓ calculates the acceleration vector correctly when $t = 0$ ✓ indicates the $\underset{\sim}{a}(0)$ vector correctly on the diagram (an arrow must be evident)

(b) Calculate the length of the path taken by the fly, correct to 0.001 metres. (2 marks)

Solution
Length of path is the distance travelled in period $0 \le t \le 2\pi$. Length of path $= \int_0^{2\pi} \left\| \underset{\sim}{v}(t) \right\| \, dt$ $= \int_0^{2\pi} \sqrt{\cos^2 t + 4\cos^2 2t + \sin^2 t} \; dt$ $= 10.540734...$ metres Hence the length of the fly's path is 10.541 metres (correct to 0.001 m)
Specific behaviours
✓ forms the correct expression for the path length (including the speed function) ✓ calculates the definite integral expression correctly

Question 17 **(7 marks)**

The shape of a decorative vase is modelled by revolving the curve AB about the y axis where $x=\sqrt{y\left(y^2-11y+35\right)}$ with $0 \le y \le 7$. All dimensions are in centimetres.



(a) Determine an integral expression, in terms of h, for the volume of water in the vase if it is filled to a depth of h cm. (2 marks)

Solution
$V(h) = \int_0^h \pi x^2 dy = \int_0^h \pi \,.\, y\left(y^2-11y+35\right) dy$
Specific behaviours
✓ writes a definite integral with correct limits with respect to y ✓ uses the correct integrand (including the factor of π)

Water is poured into the initially empty vase at a constant rate of 50 cm^3/s.

(b) Determine the time taken to fill the vase to a depth of 6 cm. (2 marks)

Solution
$V(6) = \int_0^6 \pi \,.\, y\left(y^2-11y+35\right) dy = 162\pi = 508.93800\ldots$ cm^3 $\therefore$ Time $= \dfrac{508.93800\ldots}{50} = 10.1787\ldots$ sec
Specific behaviours
✓ evaluates $V(6)$ correctly ✓ calculates the time taken correctly

With the depth at 6 cm, another 30 cm[3] of water is added to the vase.

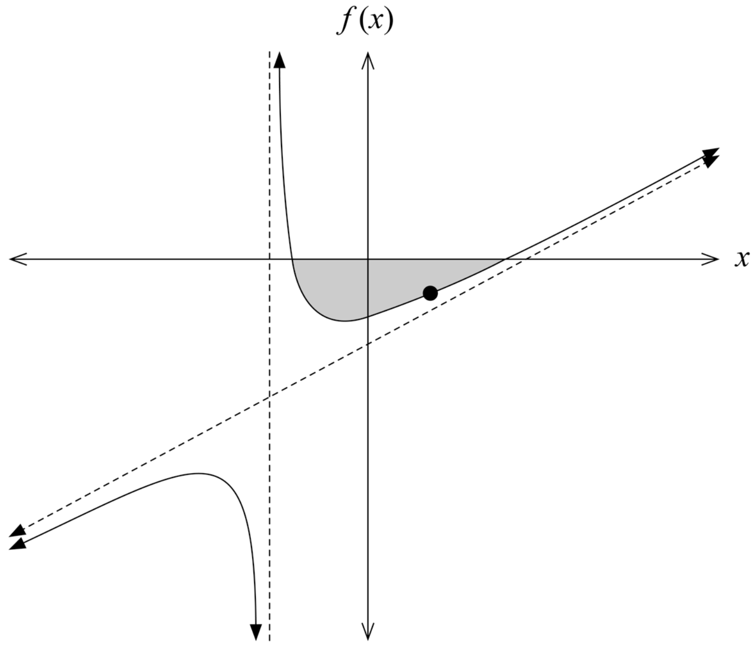
(c) Using the increments formula, calculate the approximate change in depth of water in the vase. (3 marks)

Solution
As $V(h) = \int_0^h \pi \, . \, y\left(y^2 - 11y + 35\right) dy$ $\therefore \quad \frac{dV}{dh} = \frac{d}{dh}\left(\int_0^h \pi \, . \, y\left(y^2 - 11y + 35\right) dy\right) = \pi\left(h\left(h^2 - 11h + 35\right)\right)$ $= \pi\left(h^3 - 11h^2 + 35h\right)$ (using the Fundamental Theorem) $\therefore \quad V'(6) = \pi(6)\left(6^2 - 11(6) + 35\right) = 94.24777...$ cm^2 $\Delta V \approx \frac{dV}{dh} \times \Delta h$ $30 = 94.24777... \times \Delta h$ $\Delta h = 0.3183... \, cm$ Hence the depth will increase by 0.32 cm when 30 cm^3 is poured into the vase.
Specific behaviours
✓ differentiates correctly to determine $V'(h)$ ✓ substitutes correctly for ΔV and $V'(6)$ into the increments formula ✓ calculates the change in depth Δh

Question 18 **(6 marks)**

Function $f(x)$ is a rational function of the form $\dfrac{x^2+bx+c}{x+d}$ with the following properties:

- $f(2)=-2$
- $f(x)$ has a vertical asymptote at $x=-3$ and another asymptote with equation $y=x-5$.



(a) Show that $b=-2,\ c=-10$ and $d=3$. (3 marks)

Solution
Vertical asymptote for f is $x=-3 \quad \therefore \quad d=3$ Inclined asymptote is $y=x-5$ i.e. For $\lvert x\rvert$ large $f(x) = (x-5) + \dfrac{e}{x+3}$ for some constant e $\therefore \ f(x) = \dfrac{(x+3)(x-5)}{x+3} + \dfrac{e}{x+3} = \dfrac{x^2-2x-15+e}{x+3}$ Using $f(2)=-2$ then $-2 = \dfrac{2^2-2(2)-15+e}{2+3}$ Solving gives $e=5$ i.e. $f(x)=\dfrac{x^2-2x-10}{x+3} \quad \therefore \ b=-2,\ c=-10$.
Specific behaviours
✓ refers to the vertical asymptote to deduce $d=3$ ✓ expresses function $f(x)$ in the form $(x-5) + \dfrac{e}{x+3}$ ✓ uses $f(2)=-2$ to deduce e and then $b,\ c$

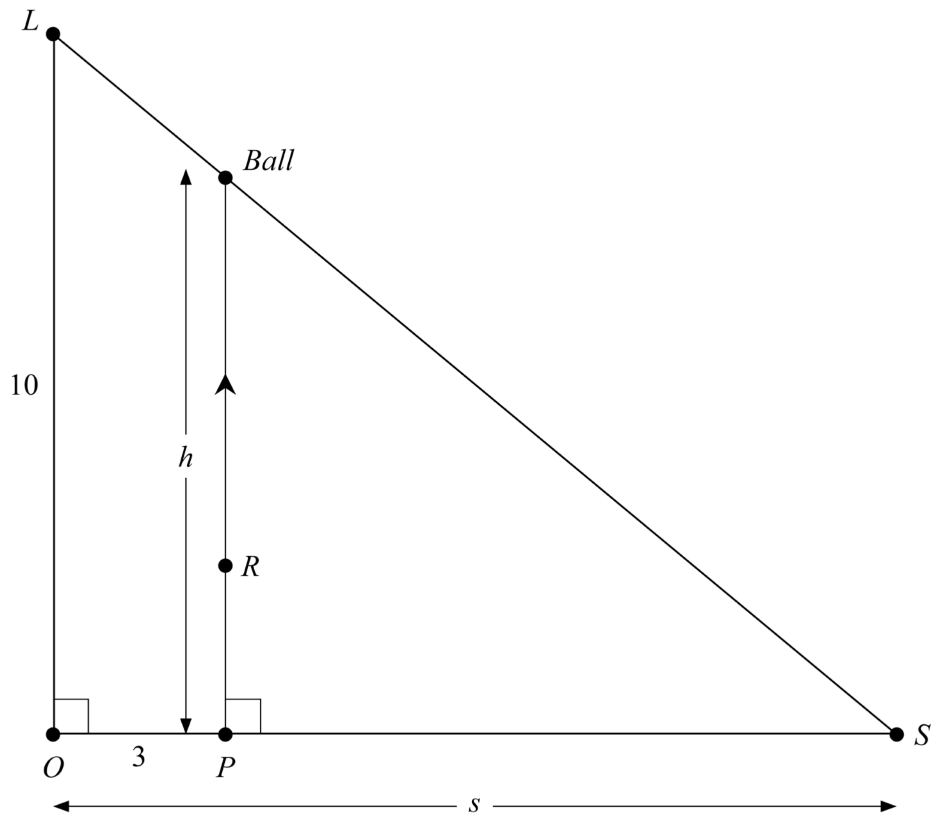
(b) Calculate the exact area of the shaded region. (3 marks)

Solution
x intercepts given by $x^2 - 2x - 10 = 0$ i.e. $x = 1 - \sqrt{11},\ \ x = 1 + \sqrt{11}$ Area shaded $= -\int_{1-\sqrt{11}}^{1+\sqrt{11}} f(x)\,dx \ = \ -\int_{1-\sqrt{11}}^{1+\sqrt{11}} \frac{x^2 - 2x - 10}{x+3}\,dx$ $= -\left(5\ln\left(\frac{4+\sqrt{11}}{4-\sqrt{11}}\right) - 8\sqrt{11}\right)$ ... from CAS $= 8\sqrt{11} - 5\ln\left(\frac{4+\sqrt{11}}{4-\sqrt{11}}\right)$ square units or $8\sqrt{11} - 5\ln\left(4+\sqrt{11}\right) + 5\ln\left(4-\sqrt{11}\right)$
Specific behaviours
✓ determines the exact x intercepts for function $f(x)$ ✓ forms the correct expression for the area in terms of a definite integral ✓ determines the expression for the exact area correctly (or its equivalent)

Question 19 **(16 marks)**

A ball is projected vertically into the air from point R, so that it will eventually hit the ground at point P, 3 metres from the base of a 10 metre high light at L.

At any time t seconds, when the ball is h metres above the ground, it casts a shadow on the ground at point S at a distance s metres from the base of the light.



At any time t it can be shown that $s(10-h) = 30$.

(a) Using implicit differentiation, show that $\dfrac{ds}{dt} = \dfrac{30}{(10-h)^2} \times \dfrac{dh}{dt}$. (3 marks)

Solution
$\dfrac{d}{dt}\left(s(10-h)\right) = \dfrac{d}{dt}(30)$ $\dfrac{ds}{dt}\times(10-h) + s(-1)\times\dfrac{dh}{dt} = 0$ $\therefore \dfrac{ds}{dt} = \dfrac{s}{(10-h)}\times\dfrac{dh}{dt}$ Substituting $s = \dfrac{30}{(10-h)}$, $\dfrac{ds}{dt} = \dfrac{30}{(10-h)}\times\dfrac{1}{(10-h)}\times\dfrac{dh}{dt} = \dfrac{30}{(10-h)^2}\times\dfrac{dh}{dt}$
Specific behaviours
✓ indicates a sum of terms equal to zero ✓ differentiates the left-hand side correctly ✓ substitutes for s correctly to obtain the desired expression for $\dfrac{ds}{dt}$

At $t = 0.5$ seconds, it is found that the ball is 6.275 metres above the ground and moving upwards at 6.1 metres per second.

(b) By assuming $h''(t) = -9.8$ ms^{-2}, show that $h(t) = 2 + 11t - 4.9t^2$. (3 marks)

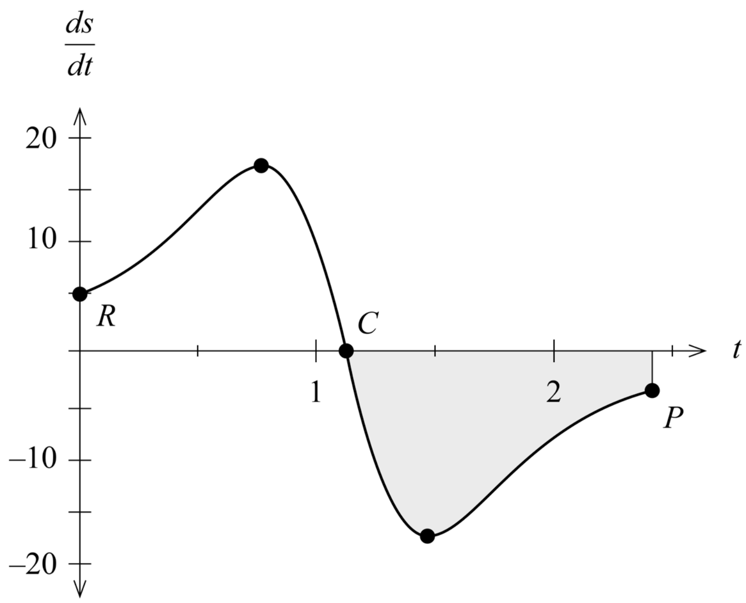
Solution
$h'(t) = \int -9.8\,dt = -9.8t + c$ Using $h'(0.5) = 6.1 \quad \therefore \quad 6.1 = -9.8(0.5) + c$ $\therefore \quad c = 11$ $h(t) = \int(-9.8t + 11)\,dt = -4.9t^2 + 11t + k$ Using $h(0.5) = 6.275 \quad \therefore \quad 6.275 = -4.9(0.5)^2 + 11(0.5) + k$ $\therefore \quad k = 2$ Hence $h(t) = 2 + 11t - 4.9t^2$ as required.
Specific behaviours
✓ anti-differentiates correctly to obtain both $h'(t)$ and $h(t)$ ✓ forms the equation correctly using $h'(0.5) = 6.1$ to solve for c ✓ forms the equation correctly using $h(0.5) = 6.275$ to solve for k

(c) Determine the initial speed of the ball's shadow, correct to the nearest 0.01 metres per second. (3 marks)

Solution
When $t = 0$, $h = 2$ and $\frac{dh}{dt} = 11 \quad (s = 3.75)$ $\therefore \quad \frac{ds}{dt} = \frac{30}{(10-2)^2} \times (11) = 5.15625$ Hence the ball's shadow initial speed is 5.16 metres per second.
Specific behaviours
✓ determines the correct values for h, $\frac{dh}{dt}$ when $t = 0$ ✓ forms the correct expression for $\frac{ds}{dt}$ using the answer from part (a) ✓ calculates the speed correctly to 0.01 metres per second

Question 19 (continued)

The graph of the function $\frac{ds}{dt}$ against time t is shown below. Point R of this graph corresponds to the ball being thrown into the air, while point P corresponds to the ball hitting the ground.



The definite integral $\int_a^b \left(\frac{ds}{dt}\right)dt$ was evaluated so that the area for the shaded region could be determined. This area is 13.4258 square units.

(d) Determine the values for a and b (correct to 0.01 seconds) and describe what this definite integral represents in terms of the motion of the shadow. (4 marks)

Solution
At point C: $\frac{ds}{dt}=0$ when $\frac{dh}{dt}=0$ i.e. $11-9.8t=0$ i.e. $t=\frac{11}{9.8}$ = $1.12244898...$ sec $\therefore$ $a=1.12$ sec   At point P: $h=0$ i.e. $2+11t-4.9t^2=0$ Solving gives $t=2.41398...$ (reject $t<0$) $\therefore$ $b=2.41$ sec $\int_{1.12}^{2.41}\left(\frac{ds}{dt}\right)dt = \Delta s = -13.4258$ m   This means that the shadow will move 13.43 metres towards the light while the ball is falling from its highest point to hitting the ground.
Specific behaviours
✓ determines the value of a correctly   ✓ determines the value of b correctly   ✓ states the shadow moves 13.43 metres back towards the light or refers to the negative change in displacement   ✓ interprets the ball is falling from its highest point to hitting the ground

(e) Determine the fastest rate at which the shadow moves (correct to the nearest 0.01 metres per second) and the time when this occurs (correct to the nearest 0.01 seconds). (3 marks)

Solution
For any time t sec: $\therefore \quad \frac{ds}{dt} = \frac{30}{(10-h)^2} \times \left(\frac{dh}{dt}\right) = \frac{30}{\left(10-\left(2+11t-4.9t^2\right)\right)^2} \times (-9.8t+11)$ Plotting the graph of the rate $r(t) = \frac{ds}{dt}$ versus t : There is a maximum at $(0.7699... , 17.4731...)$. The ball's shadow is moving at the fastest rate at $t = 0.77$ seconds. The fastest speed for the shadow is 17.47 metres per second.
Specific behaviours
✓ defines the ball's speed function $\frac{ds}{dt}$ correctly as a function of t ✓ states the time for the greatest speed ✓ states the greatest speed correct to 0.01 metres per second

Question 19 (continued)

Alternative Solution
For a maximum $\frac{ds}{dt}$ we require $\frac{d^2s}{dt^2} = 0$. $\frac{d^2s}{dt^2} = \frac{d}{dt}\left(\frac{30}{(10-h)^2}\times\frac{dh}{dt}\right) = \frac{60}{(10-h)^3}\times\left(\frac{dh}{dt}\right)^2 + \frac{30}{(10-h)^2}\times\frac{d^2h}{dt^2}$ $= \frac{60}{(10-h)^3}\times(11-9.8t)^2 + \frac{30}{(10-h)^2}\times(-9.8)$ $0 = \frac{30}{(10-h)^3}\left[2(11-9.8t)^2 - 9.8(10-h)\right]$ i.e. Solve $2(11-9.8t)^2 = 9.8\left(10-2-11t+4.9t^2\right)$ Solving gives $t = 0.76995...$ or $t = 1.47494...$ But $t < \frac{11}{9.8} = 1.122$ s The ball's shadow is moving at the fastest rate at $t = 0.77$ seconds. Hence the maximum value $s'(0.76995..) = 17.4731...$ m/s. The fastest speed for the shadow is 17.47 metres per second.
Specific behaviours
✓ differentiates the rate function $\frac{ds}{dt}$ correctly to consider $\frac{d^2s}{dt^2} = 0$ ✓ states the time for the greatest speed ✓ states the greatest speed correct to 0.01 metres per second

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Published by the School Curriculum and Standards Authority of Western Australia
303 Sevenoaks Street
CANNINGTON WA 6107