Government of **Western Australia**
**School Curriculum and Standards Authority**

# MATHEMATICS METHODS

# Calculator-free

# ATAR course examination 2017

# Marking Key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

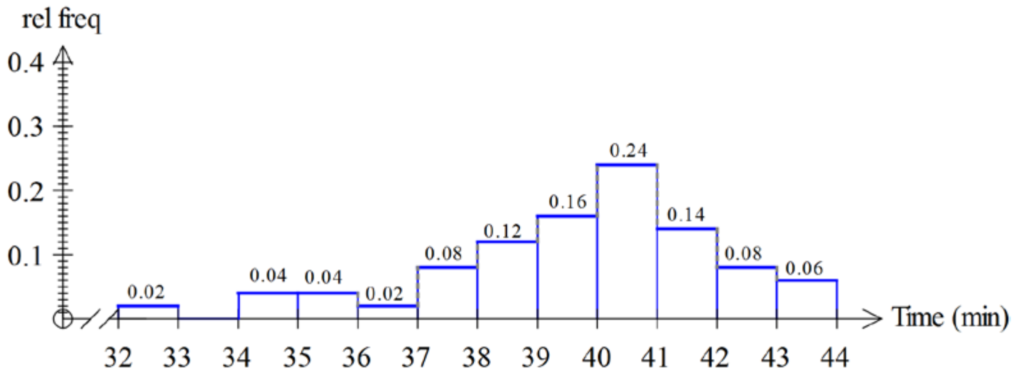


2017/69887

**Section One: Calculator-free** **35% (52 Marks)**

**Question 1** **(5 marks)**

Anastasia is a university student. She records the time it takes for her to get from home to her campus each day. The histogram of relative frequencies below shows the journey times she recorded.




Use the above data to estimate the probability of her next journey from home to her university campus

(a) taking her less than 36 minutes. (1 mark)

| Solution |
|---|
| $P(T \leq 36) = 0.02 + 0.04 + 0.04$<br>$= 0.1$ |
| **Specific behaviours** |
| ✓ sums relative frequencies to determine probability |

(b) taking at least 35 minutes but no more than 39 minutes. (2 marks)

| Solution |
|---|
| $P(35 \leq T \leq 39) = 0.04 + 0.02 + 0.08 + 0.12$<br>$= 0.26$ |
| **Specific behaviours** |
| ✓ recognises the probability involves frequencies above 35 and below 39<br>✓ sums relative frequencies to determine probability |

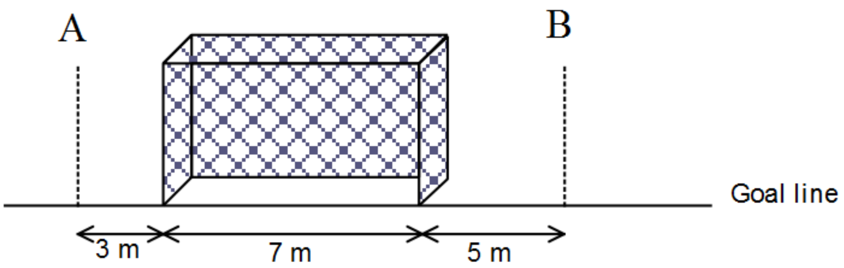
On three consecutive days, Anastasia needs to be on campus no later than 10 am.

(c) If she leaves her home at 9.22 am each day, use the above data to estimate the probability that she makes it on or before time on all three days. (2 marks)

| Solution |
|---|
| $P(T \leq 38) = 0.02 + 0.04 + 0.04 + 0.02 + 0.08$<br>$= 0.2$<br>3 consecutive days $= 0.2^3$<br>$= 0.008$ |
| **Specific behaviours** |
| ✓ sums relative frequencies to determine probability<br>✓ determines probability of 3 consecutive days |

**Question 2** **(6 marks)**

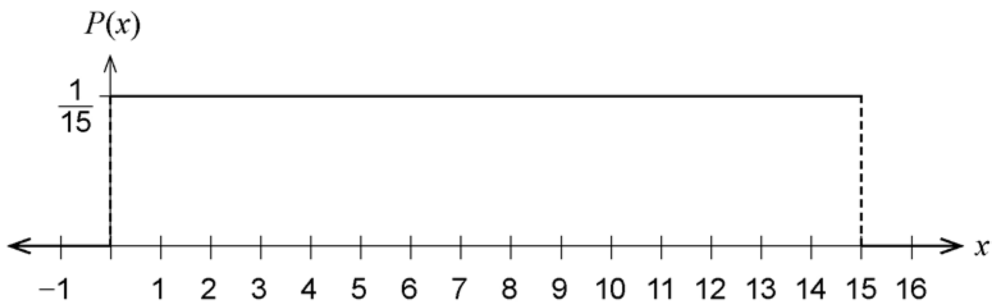
Michelle is a soccer goalkeeper and has built a machine to help her practise. The machine will shoot a soccer ball randomly along the ground at or near a goal that is seven metres wide. The machine is equally likely to shoot the ball so that the centre of the ball crosses the goal line anywhere between point A three metres left of the goal, and point B five metres right of the goal, as shown in the diagram below.




Michelle sets up a trial run without anyone in the goals. Assume the goal posts are of negligible width.

Let the random variable $X$ be the distance the centre of the ball crosses the goal line to the right of point A.

(a) Complete the graphical representation of the probability density function for the random variable $X$. (2 marks)

| Solution |
|---|
|  |
| **Specific behaviours** |
| ✓ draws uniform distribution correctly<br>✓ includes $P(x)$ axes scale |



(b) What is the probability that the machine shoots a ball so that its centre misses the goal to the left? (1 mark)

| Solution |
|---|
| $\frac{3}{15}$ |
| **Specific behaviours** |
| ✓ states correct probability |

**Question 2** (continued)

(c) What is the probability that the machine shoots a ball so that its centre is inside the goal? (1 mark)

| Solution |
|---|
| $\frac{7}{15}$ |
| **Specific behaviours** |
| ✓ states correct probability |

(d) If the machine shoots a ball so that its centre misses the goal, what is the probability that the ball's centre misses to the right? (2 marks)

| Solution |
|---|
| $\frac{\frac{5}{15}}{\frac{8}{15}} = \frac{5}{8}$ |
| **Specific behaviours** |
| ✓ correctly determines numerator<br>✓ correctly determines denominator |

**Question 3** **(4 marks)**

Solve $4e^{2x} = 81 - 5e^{2x}$ exactly for $x$.

| Solution |
|---|
| $4e^{2x} = 81 - 5e^{2x}$<br>$9e^{2x} = 81$<br>$e^{2x} = 9$<br>$\ln(e^{2x}) = \ln(9)$<br>$2x = \ln(9)$<br>$x = \dfrac{\ln(9)}{2}$ |
| **Specific behaviours** |
| ✓ collects exponential terms<br>✓ uses natural logs to simplify the equation<br>✓ uses log laws to simplify LHS of equation<br>✓ solves exactly for $x$ |

**Question 4** **(3 marks)**

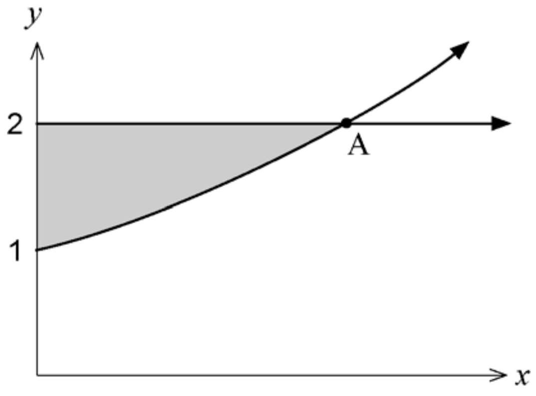
Two independent samples of different sizes were taken from a population. The first sample had sample size $n_1$ and the second sample had sample size $n_2$. The sample proportions of males in the samples were the same. When 99% confidence intervals were calculated for each sample, it was found that the corresponding margin of error in the second sample was half that of the first sample.

What is the ratio of the two sample sizes, $\dfrac{n_2}{n_1}$?

| Solution |
|---|
| $z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n_2}} = \dfrac{1}{2} z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n_1}}$<br>$\dfrac{\hat{p}(1-\hat{p})}{n_2} = \dfrac{1}{4} \times \dfrac{\hat{p}(1-\hat{p})}{n_1}$<br>$\dfrac{n_2}{n_1} = 4$ |
| **Specific behaviours** |
| ✓ uses formula for margin of error to relate the two sample sizes<br>✓ simplifies equation by squaring both sides and cancelling<br>✓ rearranges to find ratio |

**Question 5** **(8 marks)**

(a) Consider the shaded area shown between the graph of $y = e^x$, the $y$ axis and the line $y = 2$.




(i) Determine the coordinates of point A. (1 mark)

| Solution |
|---|
| $2 = e^x$<br>$x = \ln 2$ Point A has coordinates $(\ln 2, 2)$ |
| **Specific behaviours** |
| ✓ determines correct coordinates |

(ii) Hence or otherwise determine the area between the graph of $y = e^x$, the $y$ axis and the line $y = 2$. (3 marks)

| Solution |
|---|
| The required area is the area of rectangle less the area between graph and $x$-axis between $x = 0$ and $x = \ln 2$<br>$2ln2 - \int_0^{ln2} e^x dx = 2ln2 - [e^x]_0^{ln2} = 2ln2 - (2-1) = 2ln2 - 1$<br>OR<br>$x = \ln y$<br>$\int_1^2 \ln y \, dy = [y \ln y - y]_1^2 = 2\ln 2 - 1$ |
| **Specific behaviours** |
| ✓ determines area of rectangle<br>✓ writes correct integral for missing area<br>✓ determines correct answer |

(b) If the area between the graph of $y = e^x$, the $y$ axis, the $x$ axis and the line $x = k$, where $k \geq 0$, is to be equal to 2 square units, determine the exact value of $k$. (4 marks)

| Solution |
|---|
| $\int_0^k e^x \, dx = 2$<br>$\therefore e^k - e^0 = 2$<br>$e^k = 3$<br>$k = \ln 3$ |
| **Specific behaviours** |
| ✓ identifies integral to determine area<br>✓ equates integral to 2<br>✓ evaluates integral<br>✓ solves for $k$ |

**Question 6** **(7 marks)**

(a) Evaluate $\int_0^1 \frac{-12x}{1+3x^2}\,dx.$ (3 marks)

| Solution |
| --- |
| $\int_0^1 \frac{-12x}{1+3x^2}\,dx = -2\ln\left[1+3x^2\right]_0^1$<br>$= -2\left(\ln(4) - \ln(1)\right)$<br>$= \ln(\frac{1}{16})$ or $\{-\ln 16\}$ or $-2\ln 4$ |
| **Specific behaviours** |
| ✓ identifies the solution involving ln<br>✓ determines correct expression<br>✓ evaluates limits and correctly determines and simplifies solution |

(b) Given $f(x) = \ln(2 - x^3)$

(i) determine $f'(1)$. (3 marks)

| Solution |
| --- |
| $\frac{dy}{dx} = \frac{-3x^2}{2-x^3}$<br>$\left.\frac{dy}{dx}\right\|_{x=1} = \frac{-3}{1}$<br>$= -3$ |
| **Specific behaviours** |
| ✓ identifies the need for the quotient rule to find derivative<br>✓ correctly determines derivative<br>✓ determines the derivative at $x$=1 |

(ii) In relation to the graph of $f(x)$, explain the meaning of your answer to (b)(i). (1 mark)

| Solution |
| --- |
| $f'(1)$ is the gradient of the curve (or the tangent to the curve) at the point where $x$=1 |
| **Specific behaviours** |
| ✓ explains meaning |

**Question 7** **(6 marks)**

Given that $\log_{10} 2 = x$ and $\log_{10} 7 = y$

(a) express $\log_{10} 14$ in terms of $x$ and $y$. (2 marks)

| Solution |
|---|
| $\log_{10} 14 = \log_{10}(2 \times 7) = \log_{10}2 + \log_{10}7 = x + y$ |
| **Specific behaviours** |
| ✓ expresses 14 as $2 \times 7$<br>✓ uses log laws to obtain the expression |

(b) show that $\log_{10} 17.5 = y - 2x + 1$. (2 marks)

| Solution |
|---|
| $\log_{10}17.5 = \log_{10}\frac{70}{4} = \log_{10}7 + \log_{10}10 - \log_{10}2^2 = y - 2x + 1$ |
| **Specific behaviours** |
| ✓ uses log laws correctly to expand<br>✓ uses the log law for a power to obtain the correct expression |

(c) evaluate $10^{y-x}$. (2 marks)

| Solution |
|---|
| $10^x = 2$<br>$10^y = 7$<br>$10^{y-x} = \frac{10^y}{10^x}$<br>$= \frac{7}{2}$ |
| **Specific behaviours** |
| ✓ rewrites logarithmic equations in exponential form<br>✓ uses index laws to evaluate |

**Question 8** **(5 marks)**

(a) Differentiate $2x\sin(3x)$ with respect to $x$. (2 marks)

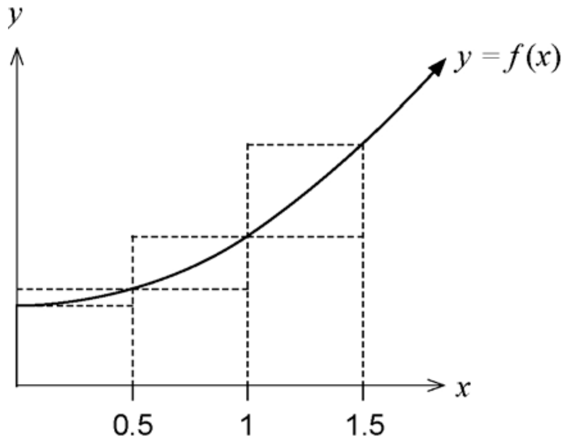
| Solution |
| --- |
| $\frac{d(2x\sin(3x))}{dx}$<br>$= 2 \times \sin(3x) + 2x \times 3\cos(3x)$<br>$= 2\sin(3x) + 6x\cos(3x)$ |
| **Specific behaviours** |
| ✓ uses product rule<br>✓ obtains correct answer |

(b) Hence show that $\int x\cos(3x)dx = \frac{3x\sin(3x)+\cos(3x)}{9} + c$. (3 marks)

| Solution |
| --- |
| $\frac{d(2x\sin(3x))}{dx} = 2\sin(3x) + 6x\cos(3x)$<br>$\int \frac{d(2x\sin(3x))}{dx}dx = \int (2\sin(3x) + 6x\cos(3x))dx$<br>$2x\sin(3x) + c_1 = \int 2\sin(3x)dx + 6\int x\cos(3x)\,dx$<br>$\frac{2x\sin(3x) + c_1}{6} = \frac{-2\cos(3x)}{18} + c_2 + \int x\cos(3x)\,dx$<br>$\int x\cos(3x)\,dx = \frac{2x\sin(3x)}{6} + \frac{2\cos(3x)}{18} + c$<br>$\therefore \int x\cos(3x)dx = \frac{3x\sin(3x)+\cos(3x)}{9} + c$ |
| **Specific behaviours** |
| ✓ integrates both sides<br>✓ uses fundamental thm to simplify LHS<br>✓ evaluates integrals and rearranges to show result |

**Question 9** **(8 marks)**

Consider the function $f(x)$ shown graphed below. The table gives the value of the function at the given $x$ values.




| $x$ | 0 | 0.5 | 1 | 1.5 |
|---|---|---|---|---|
| $f(x)$ | 20 | 21 | 24 | 29 |

(a) By considering the areas of the rectangles shown, demonstrate and explain why $32.5 < \int_0^{1.5} f(x)\,dx < 37$.

(3 marks)

| **Solution** |
|---|
| lower limit $= 20\times0.5+21\times0.5+24\times0.5$ <br> $=10+10.5+12$ <br> $=32.5$ <br> upper limit $= 21\times0.5+24\times0.5+29\times0.5$ <br> $=10.5+12+14.5$ <br> $=37$ <br> therefore $=\int_0^{1.5} f(x)\,dx$ is between these values as this is the area under the curve |
| **Specific behaviours** |
| ✓ shows a calculation to produce an underestimate of area <br> ✓ shows a calculation to produce an overestimate of area <br> ✓ explains the limits in terms of area |

**Question 9** (continued)

Consider the table of further values of $f(x)$ given below.

| $x$ | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 20 | 21 | 24 | 29 | 36 | 45 | 56 |

(b) Use the table values to determine the best estimate possible for $\int_1^3 f(x)dx$. (3 marks)

| Solution |
|---|
| New under and over are:<br>under est $= 24\times0.5+29\times0.5+36\times0.5+45\times0.5$<br>$=67$<br>over est $= 29\times0.5+36\times0.5+45\times0.5+56\times0.5$<br>$=83$<br>area $\approx \dfrac{67+83}{2}$<br>$=75$ |
| **Specific behaviours** |
| ✓ determines the under estimate<br>✓ determines the over estimate<br>✓ finds the mean to produce best estimate of area |

(c) State two ways in which you could determine a more accurate value for $\int_1^3 f(x)dx$.

(2 marks)

| Solution |
|---|
| • by reducing the width of the rectangles and, therefore, using more rectangles to estimate the area the error in the estimate would be reduced<br>• determining the function and using calculus |
| **Specific behaviours** |
| ✓ states one reason<br>✓ states two reasons |

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