Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS SPECIALIST

Calculator-free

ATAR course examination 2017

Marking Key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2017/69889

Section One: Calculator-free **35% (53 marks)**

Question 1 **(4 marks)**

Let $z = a - bi$, where $a > 0,\ b > 0$. Consider $w = z + i\bar{z}$.

Determine the possible value(s) for $\arg(w)$.

Solution
$w = (a-bi) + i(a+bi)$ $= a - bi + ai + bi^2$ $= a - bi + ai - b$ $= (a-b) + (a-b)i$ $\therefore \ \operatorname{Re}(w) = \operatorname{Im}(w)$ As $a-b$ could be either positive or negative then $\arg(w) = \frac{\pi}{4}$ or $-\frac{3\pi}{4}$. Alternatively expressed $\arg(w) = \frac{\pi}{4} \pm n\pi$.
Specific behaviours
✓ uses the correct expression for $\bar{z}$ in terms of a, b ✓ determines the correct expression for w in terms of a, b ✓ states that $\arg(w) = \frac{\pi}{4}$ ✓ states that $\arg(w) = -\frac{3\pi}{4}$ (permit $\arg(w) = \frac{5\pi}{4}$)

Question 2 **(6 marks)**

Consider $f(z) = 2z^3 - 5z^2 + 4z - 10$ where z is a complex number.

(a) Show that $\left(z - \sqrt{2}i\right)$ is a factor of $f(z)$. (2 marks)

Solution
$f\left(\sqrt{2}i\right) = 2\left(\sqrt{2}i\right)^3 - 5\left(\sqrt{2}i\right)^2 + 4\left(\sqrt{2}i\right) - 10$ $= -4\sqrt{2}i + 10 + 4\sqrt{2}i - 10$ $= 0$ Hence $\left(z - \sqrt{2}i\right)$ is a factor of $f(z)$.
Specific behaviours
✓ substitutes $z = \sqrt{2}i$ correctly ✓ provides evidence that $f\left(\sqrt{2}i\right) = 0$ i.e. not just the statement $f\left(\sqrt{2}i\right) = 0$

(b) Given that $\left(z - \sqrt{2}i\right)$ is a factor of $f(z)$, state another factor of $f(z)$. (1 mark)

Solution
Since $\left(z - \sqrt{2}i\right)$ is a factor of $f(z)$, then the conjugate factor $\left(z + \sqrt{2}i\right)$ will be also.
Specific behaviours
✓ states the conjugate factor (or the factor $2z - 5$)

(c) Solve the equation $2z^3 - 5z^2 + 4z - 10 = 0$. (3 marks)

Solution
We know that both $\left(z - \sqrt{2}i\right)$ and $\left(z + \sqrt{2}i\right)$ are factors. $\therefore\ f(z) = \left(z - \sqrt{2}i\right)\left(z + \sqrt{2}i\right)Q(x)$ where $Q(x) = ax + b$ $= \left(z^2 + 2\right)\left(2z - 5\right)$ Hence to solve $2z^3 - 5z^2 + 4z - 10 = 0$ $\therefore\ \left(z^2 + 2\right)\left(2z - 5\right) = 0$ i.e. $z = \sqrt{2}i,\ -\sqrt{2}i,\ \frac{5}{2}$.
Specific behaviours
✓ factorises $f(z)$ correctly ✓ states the solutions $z = \pm\sqrt{2}i$ i.e. two solutions ✓ states the solution $z = \frac{5}{2}$ i.e. all three solutions

Question 3 **(7 marks)**

Consider the definite integral $\int_0^1 \frac{x^2}{\left(1+x^2\right)^2}\,dx$.

(a) By using the substitution $x = \tan u$, show that $\int_0^1 \frac{x^2}{\left(1+x^2\right)^2}\,dx = \int_a^b \sin^2 u\,du$ and state the values of a, b. (4 marks)

Solution

When $x = 0, u = 0$ and $x = 1, u = \frac{\pi}{4}$ $\quad \frac{dx}{du} = \sec^2 u \quad \therefore\ dx = \sec^2 u\,du$

$$\int_0^1 \frac{x^2}{\left(1+x^2\right)^2}\,dx = \int_0^{\frac{\pi}{4}} \frac{\tan^2 u}{\left(1+\tan^2 u\right)^2}.\sec^2 u\,du$$

$$= \int_0^{\frac{\pi}{4}} \frac{\tan^2 u \sec^2 u}{\left(\sec^2 u\right)^2}\,du = \int_0^{\frac{\pi}{4}} \frac{\tan^2 u}{\sec^2 u}\,du = \int_0^{\frac{\pi}{4}} \frac{\sin^2 u}{\cos^2 u} \times \frac{\cos^2 u}{1}\,du$$

$$= \int_0^{\frac{\pi}{4}} \sin^2 u\,du$$

Specific behaviours

- ✓ changes the limits correctly i.e. determines the correct values for a, b
- ✓ differentiates $\tan u$ correctly to determine dx in terms of du
- ✓ substitutes for $1+\tan^2 u$ correctly using the trigonometric identity
- ✓ expresses $\tan u$ and $\sec u$ in terms of $\sin u, \cos u$ correctly

(b) Hence evaluate $\int_0^1 \frac{x^2}{\left(1+x^2\right)^2}\,dx$ exactly. (3 marks)

Solution

$$\int_0^1 \frac{x^2}{\left(1+x^2\right)^2}\,dx = \int_0^{\frac{\pi}{4}} \sin^2 u\,du = \int_0^{\frac{\pi}{4}} \frac{1}{2}\left(1-\cos 2u\right)du = \left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{\frac{\pi}{4}}$$

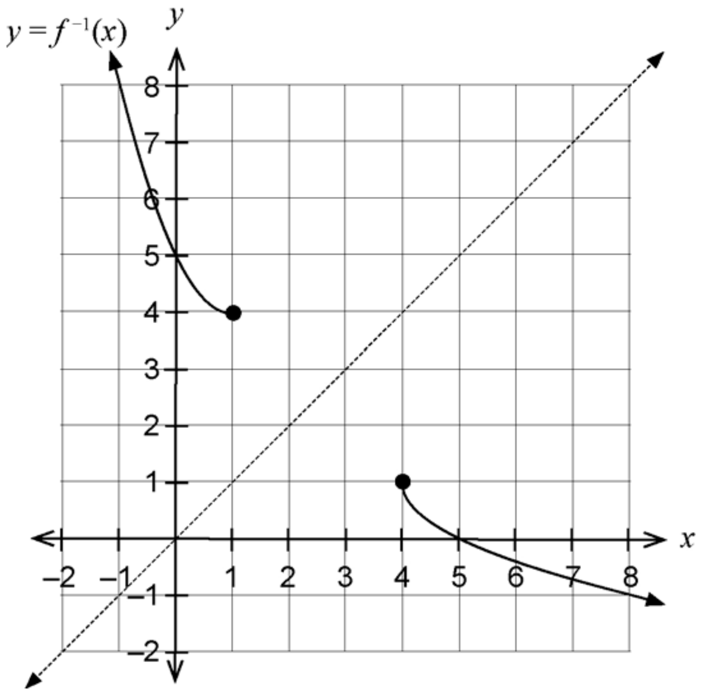
$$= \frac{\pi}{8} - \frac{1}{4}$$

Specific behaviours

- ✓ expresses the integrand correctly using the cosine double angle identity
- ✓ anti-differentiates correctly
- ✓ evaluates correctly using an exact value

Question 4 **(9 marks)**

Function f is defined as $f(x) = 1 - \sqrt{x-4}$. The graph of $y = f(x)$ is shown below.



(a) Sketch the graph of $y = f^{-1}(x)$ on the axes above. (2 marks)

Solution
As shown above.
Specific behaviours
✓ reflects the graph of $y = f(x)$ about the line $y = x$ ✓ contains the points $(0,5)$ and $(-1,8)$

(b) Determine the defining rule for $y = f^{-1}(x)$ and state its domain. (3 marks)

Solution
$f: \quad y = 1 - \sqrt{x-4} \qquad R_f = \{y \mid y \le 1\}$ $f^{-1}: \quad x = 1 - \sqrt{y-4}$ $\therefore \quad \sqrt{y-4} = 1 - x$ $\therefore \quad y - 4 = (1-x)^2 \qquad \therefore \quad f^{-1}(x) = (1-x)^2 + 4, \qquad D_{f^{-1}} = \{x \mid x \le 1\} = R_f$
Specific behaviours
✓ interchanges x, y to write the rule for the inverse ✓ obtains the correct defining rule for $y = f^{-1}(x)$ ✓ states the correct domain for $y = f^{-1}(x)$

Question 4 (continued)

Function g is defined as $g(x)=\frac{1}{x^2}$.

(c) Determine an expression for $f\circ g(x)$. (1 mark)

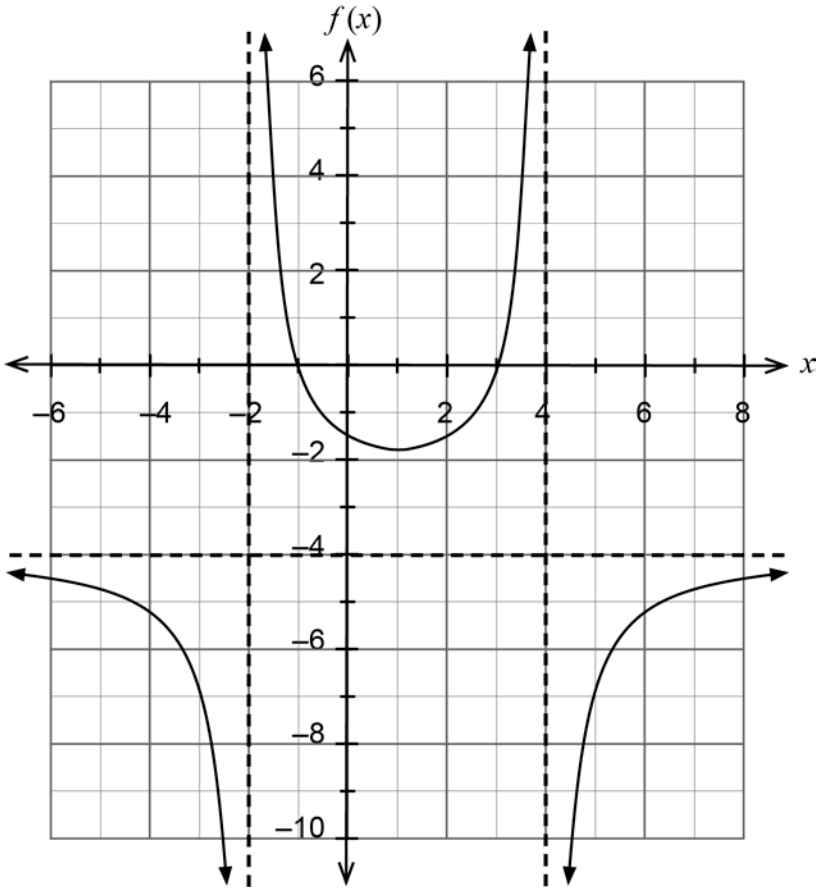
Solution
$f \circ g(x) = f\left(\frac{1}{x^2}\right) = 1-\sqrt{\frac{1}{x^2}-4}$
Specific behaviours
✓ writes the correct expression for $f\circ g(x)$ (no simplification required)

(d) For $f\circ g(x)$, determine the domain. (3 marks)

Solution
We require $x^2 \neq 0$ so $g(x)$ is defined and $\frac{1}{x^2}-4 \geq 0$ so the square root is defined. Solving $\frac{1}{x^2}-4 \geq 0$ yields $x^2 \leq \frac{1}{4}$ i.e. $-\frac{1}{2}\leq x\leq\frac{1}{2}$ Hence $D_{fog}=\{x\mid -\frac{1}{2}\leq x\leq \frac{1}{2},\ x\neq 0\}$
Specific behaviours
✓ states that $\frac{1}{x^2}-4 \geq 0$ i.e. square root operation will be defined ✓ states that $-\frac{1}{2}\leq x\leq\frac{1}{2}$ ✓ states that $x\neq 0$

Question 5 **(6 marks)**

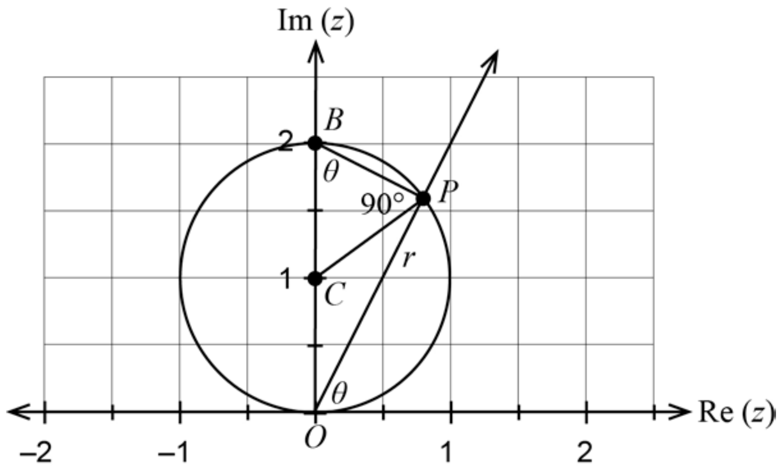
Sketch the graph of $f(x) = -\dfrac{4(x-3)(x+1)}{x^2-2x-8}$ on the axes below.



Solution
Shown above.
Specific behaviours
✓ indicates x intercepts at $x=-1$ and $x=3$ ✓ indicates vertical asymptotes at $x=-2$ and $x=4$ ✓ indicates a turning point at $x=1$ (midway between the x intercepts) ✓ indicates EITHER the correct position for the vertical intercept $\left(0,\ -\dfrac{3}{2}\right)$ OR the local minimum $\left(1,\ -\dfrac{16}{9}\right)$ ✓ indicates a horizontal asymptote at $y=-4$ ✓ indicates the correct curvature either side of the asymptotes

Question 6 **(6 marks)**

A circle and a ray are indicated in the complex plane. The ray has equation $\arg(z) = \tan^{-1}(2)$. Point C is the centre of the circle. Point P is the intersection of the circle and the ray.



(a) Determine the equation for the circle. (2 marks)

Solution
Equation of the circle is $\lvert z-i \rvert = 1$
Specific behaviours
✓ writes the modulus expression correctly (correct centre) ✓ writes the correct constant (radius)

Point P determines a complex number $w = r\,cis\,\theta$.

(b) Determine the exact values for r, θ. (4 marks)

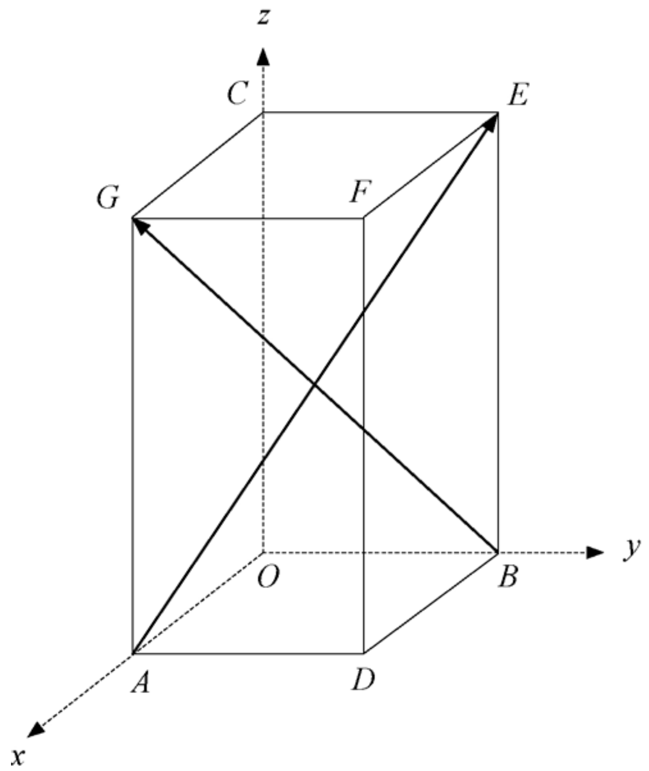
Solution
We know that $\theta = \arctan(2)$ or $\theta = \tan^{-1}(2)$ Solving simultaneously to determine $P = x + yi$ Circle $x^2 + (y-1)^2 = 1$ Ray $y = 2x$ Substituting $y = 2x$: $x^2 + (2x-1)^2 = 1$ i.e. $5x^2 - 4x = 0$ Solving gives $x = \frac{4}{5},\ y = \frac{8}{5}$ $r^2 = x^2 + y^2 = \left(\frac{4}{5}\right)^2 + \left(\frac{8}{5}\right)^2 = \frac{80}{25} \quad \therefore \ r = \frac{4\sqrt{5}}{5}$
Specific behaviours
✓ states the correct value for θ ✓ forms cartesian equations correctly to solve simultaneously ✓ solves for x, y correctly to determine P ✓ determines the correct value for r

Alternative Solution 1
$\theta = \arctan(2) = s\angle OBP$ since $s\angle BPO = 90^o$ Angle in a semi-circle theorem In right ΔBPO $\tan(\theta) = \frac{OP}{BP} = \frac{r}{\sqrt{2^2 - r^2}}$ i.e. $2 = \frac{r}{\sqrt{4-r^2}}$ i.e. $4 = \frac{r^2}{4-r^2}$ $\therefore$ $16 - 4r^2 = r^2$ i.e. $r^2 = \frac{16}{5}$ $\therefore$ $r = \frac{4}{\sqrt{5}}$
Specific behaviours
✓ states the correct value for θ ✓ identifies ΔBPO is a right triangle (geometric properties) ✓ forms a correct equation to solve for r ✓ determines the correct value for r

Alternative Solution 2
$\theta = \arctan(2)$ In isosceles ΔCPO $s\angle OCP = 2\theta = 2\tan^{-1}(2)$ Central angle theorem Using the cosine rule: $r^2 = 1^2 + 1^2 - 2(1)(1)\cos 2\theta$ i.e. $r = \sqrt{2 - 2\cos\left(2\tan^{-1}(2)\right)}$
Specific behaviours
✓ states the correct value for θ ✓ identifies ΔBPO as isosceles with $s\angle OCP = 2\theta$ (geometric properties) ✓ forms a correct equation to solve for r ✓ determines a correct expression for r

Question 7 **(10 marks)**

A right rectangular prism, with square base $OADB$, is shown below. Point O is the origin and points A, B, C have respective position vectors $\begin{pmatrix}4\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\4\\0\end{pmatrix}$, $\begin{pmatrix}0\\0\\c\end{pmatrix}$ where $c > 0$.



(a) Determine, in terms of c, the:

(i) vector equation for the line containing points A and E. (3 marks)

Solution
Direction for $\overrightarrow{AE} = \overrightarrow{AD} + \overrightarrow{DB} + \overrightarrow{BE} = \begin{pmatrix}0\\4\\0\end{pmatrix} + \begin{pmatrix}-4\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\c\end{pmatrix} = \begin{pmatrix}-4\\4\\c\end{pmatrix}$ Equation for $\overleftrightarrow{AE}$: $\underset{\sim}{r} = \begin{pmatrix}4\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}-4\\4\\c\end{pmatrix} = \begin{pmatrix}4-4\lambda\\4\lambda\\c\lambda\end{pmatrix}$
Specific behaviours
✓ determines the direction vector for the line correctly   ✓ uses the position vector for a known point on the line correctly   ✓ forms the vector equation for the line correctly using a parameter

(ii) cartesian equation for the plane $ADEC$. (4 marks)

Solution
From the vector equation of the form : $\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}$
$\underset{\sim}{n} = \overrightarrow{AD} \times \overrightarrow{AC} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \times \begin{pmatrix} -4 \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} 4c - 0(0) \\ 0(-4) - 0(c) \\ 0(0) - 4(-4) \end{pmatrix} = \begin{pmatrix} 4c \\ 0 \\ 16 \end{pmatrix}$
i.e. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 4c \\ 0 \\ 16 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4c \\ 0 \\ 16 \end{pmatrix}$ i.e. $4cx + 16z = 16c$ $cx + 4z = 4c \quad y \in \mathbb{R}$
Specific behaviours
✓ writes correct expressions for vectors in the plane   ✓ determines the normal vector for the plane correctly   ✓ forms the vector equation for the plane correctly   ✓ determines the correct cartesian equation in terms of c

Alternative Solution
From the vector equation of the form :
$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\left(\overrightarrow{AD}\right) + \mu\left(\overrightarrow{AC}\right) = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} 4 - 4\mu \\ 4\lambda \\ c\mu \end{pmatrix}$
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 - 4\mu \\ 4\lambda \\ c\mu \end{pmatrix}$ i.e. $\mu = \frac{4 - x}{4} \quad \therefore \quad z = c\left(\frac{4 - x}{4}\right)$
i.e. $\therefore \quad 4z = c(4 - x)$ $\therefore \quad cx + 4z = 4c \qquad y \in \mathbb{R}$
Specific behaviours
✓ writes correct expressions for vectors in the plane   ✓ forms the vector equation of the plane correctly   ✓ eliminates a parameter to relate the cartesian coordinates   ✓ determines the correct cartesian equation in terms of c

Question 7 (continued)

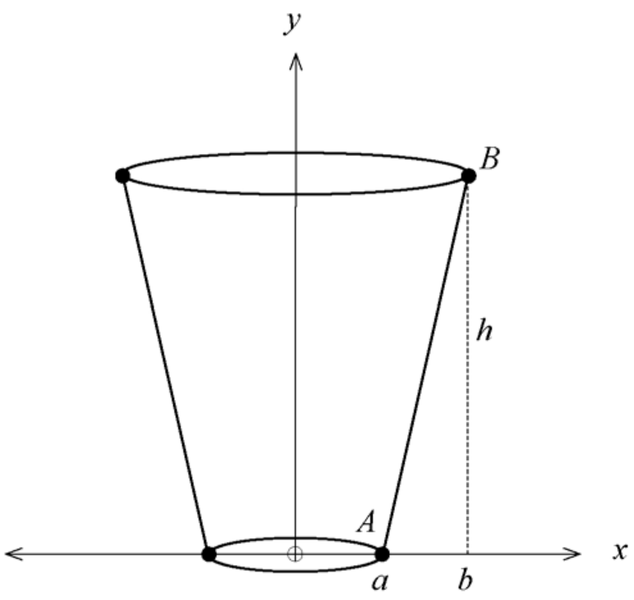
In general, the main diagonals $\overrightarrow{AE}, \overrightarrow{BG}$ are not perpendicular to each other.

(b) Determine the value of c so that the main diagonals of the prism are perpendicular to each other. (3 marks)

Solution
Main diagonals are perpendicular if $\overrightarrow{AE} \cdot \overrightarrow{BG} = 0$ i.e. $\begin{pmatrix} -4 \\ 4 \\ c \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -4 \\ c \end{pmatrix} = 0$ $\therefore \quad -16-16+c^2 = 0$ $\therefore \quad c^2 = 32$ $\therefore \quad c = \sqrt{32} = 4\sqrt{2}$
Specific behaviours
✓ writes expressions for the vectors $\overrightarrow{AE}, \overrightarrow{BG}$ correctly   ✓ forms the equation that the dot product must be zero   ✓ solves correctly to determine the value of c

Question 8 **(5 marks)**

The inner surface of a drinking glass can be modelled by rotating the line segment $\overline{AB}$ about the y axis, as shown in the diagram below. The radius of the glass at the bottom is a cm and the radius at the top is b cm. The height of the glass is h cm.



The equation for $\overline{AB}$ is $y = \left(\frac{x-a}{b-a}\right)h$.

(a) Write an expression, in terms of a definite integral, for the volume of liquid contained by the glass when it is full. (2 marks)

Solution
From $y = \left(\frac{x-a}{b-a}\right)h$, we can express $x = \left(\frac{b-a}{h}\right)y + a$. Volume $V = \int_0^h \pi (x)^2 \, dy$ $= \int_0^h \pi \left(\left(\frac{b-a}{h}\right)y + a\right)^2 dy$
Specific behaviours
✓ expresses the x coordinate correctly in terms of y ✓ writes the definite integral correctly

Question 8 (continued)

(b) By using an anti-derivative, obtain a simplified expression/formula (in terms of a, b and h) for the volume of liquid contained by the glass when it is full. (3 marks)

Solution
$V = \int_0^h \pi\left(\left(\frac{b-a}{h}\right)y+a\right)^2 dy$ $= \pi\left[\frac{\left(\left(\frac{b-a}{h}\right)y+a\right)^3}{3}\times\frac{h}{(b-a)}\right]_{y=0}^{y=h}$ $= \frac{\pi h}{3(b-a)}\left[\left(\left(\frac{b-a}{h}\right)y+a\right)^3\right]_{y=0}^{y=h}$ $= \frac{\pi h}{3(b-a)}\left[\left(\left(\frac{b-a}{h}\right)h+a\right)^3-\left(\left(\frac{b-a}{h}\right)0+a\right)^3\right]$ $= \frac{\pi h}{3(b-a)}\left[b^3-a^3\right] \quad = \quad \frac{\pi h\left(b^3-a^3\right)}{3(b-a)} \quad \ldots \ (1)$ $= \frac{\pi h}{3(b-a)}\left[(b-a)\left(b^2+ab+a^2\right)\right]$ $= \frac{\pi h}{3}\left(b^2+ab+a^2\right)$
Specific behaviours
✓ writes the anti-derivative of the quadratic term correctly   ✓ multiplies by the factor $\frac{h}{b-a}$ correctly i.e. considers the chain rule   ✓ simplifies to the expression at the line marked (1) or develops further

Alternative Solution
$V = \int_0^h \pi\left(\left(\frac{b-a}{h}\right)y + a\right)^2 dy \;=\; \pi\int_0^h \left(\left(\frac{b-a}{h}\right)^2 y^2 + 2a\left(\frac{b-a}{h}\right)y + a^2\right) dy$ $= \pi\left[\left(\frac{b-a}{h}\right)^2 \frac{y^3}{3} + a\left(\frac{b-a}{h}\right)y^2 + a^2 y\right]_{y=0}^{y=h}$ $= \pi\left[\frac{(b-a)^2}{h^2}\frac{h^3}{3} + a\left(\frac{b-a}{h}\right)h^2 + a^2 h\right]$ $= \pi\left[\left(\frac{(b-a)^2 h}{3}\right) + ah(b-a) + a^2 h\right]$ $= \frac{\pi h}{3}\left[b^2 - 2ab + a^2 + 3ab - 3a^2 + 3a^2\right]$ $= \frac{\pi h}{3}\left(b^2 + ab + a^2\right)$ ..... (1)
Specific behaviours
✓ expands the integrand correctly ✓ anti-differentiates term by term correctly ✓ simplifies to the expression at the line marked (1)

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