Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS METHODS

Calculator-assumed

ATAR course examination 2019

Marking key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2019/63005

Section Two: Calculator-assumed **65% (99 Marks)**

Question 8 **(7 marks)**

Big Foods is a large supermarket company. The manager of Big Foods wants to estimate the proportion of households that do the majority of their grocery shopping in their stores.

A junior staff member at Big Foods conducted a survey of 250 randomly-selected households and found that 56 did the majority of their grocery shopping at a Big Foods store.

(a) (i) Calculate the sample proportion of households who did the majority of their grocery shopping at Big Foods. (1 mark)

Solution
$\hat{p} = \frac{56}{250}$ $= 0.224$
Specific behaviours
✓ calculates correct proportion

(ii) Determine the 95% confidence interval for the proportion of households who do the majority of their grocery shopping at Big Foods. Give your answer to four decimal places. (3 marks)

Solution
$\left(0.224 - 1.96\sqrt{\frac{0.224(1-0.224)}{250}},\ 0.224 + 1.96\sqrt{\frac{0.224(1-0.224)}{250}}\right)$ $(0.1723, 0.2757)$
Specific behaviours
✓ uses $Z = 1.96$ ✓ calculates confidence interval   ✓ rounds to four decimal places

(iii) What is the margin of error of the 95% confidence interval? Give your answer to four decimal places. (1 mark)

Solution
Either $E = 1.96\sqrt{\frac{0.224(1-0.224)}{250}}$ $= 0.0517$ or $E = \frac{0.2757 - 0.1723}{2}$ $= 0.0517$
Specific behaviours
✓ calculates margin of error

An independent survey company conducted a large-scale survey of household supermarket preferences and estimated that the true proportion of households that conduct most of their grocery shopping at Big Foods was 0.17 (assume that this is indeed the true proportion).

(b) With reference to your answer to part (a)(ii), does this result suggest that the junior staff member at Big Foods made a mistake? (2 marks)

Solution
No. Only 95% of 95% confidence intervals are expected to contain the true proportion. It is possible that the survey and calculation by the junior staff member was performed appropriately, but happened to yield one of the 5% of confidence intervals that do not contain the true proportion.
Specific behaviours
✓ answers ‘No’ with a reference to part (a) ✓ justifies answer by saying that only 95% of intervals are expected to contain the true proportion

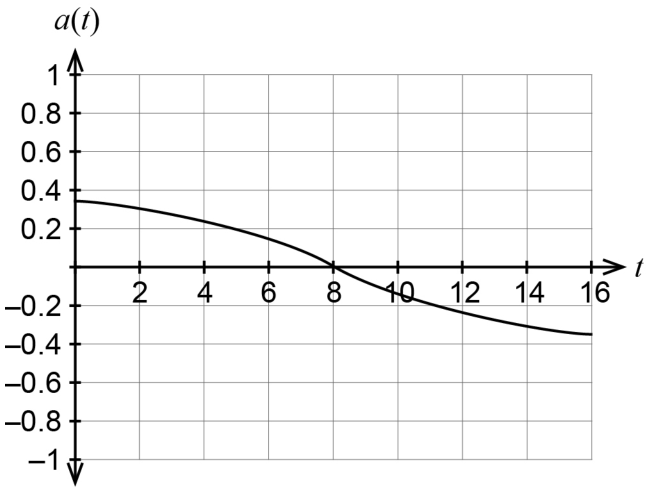
Question 9 **(8 marks)**

It takes an elevator 16 seconds to ascend from the ground floor of a building to the sixth floor. The velocity of the elevator during its ascent is given by

$$v(t) = \frac{9\pi}{16}\sin\left(\frac{\pi t}{16}\right) \text{ m/s.}$$

The velocityis measured in metres per second, while the time, t, is measured in seconds.

(a) Determine the acceleration of the elevator during its ascent and provide a sketch of the acceleration function for $0 \leq t \leq 16$. (2 marks)

Solution
 $a(t) = \frac{9\pi^2}{256}\cos\left(\frac{\pi t}{16}\right)$ m^2/s
Specific behaviours
✓ differentiates the velocity function to obtain the acceleration ✓ provides a sketch clearly showing the peak acceleration values and correctly locating the zero acceleration point

(b) With reference to your answer from part (a), explain what is happening to the velocity of the elevator in the interval $0 < t < 8$ and in the interval $8 < t < 16$. (3 marks)

Solution
Since the acceleration is positive on the interval $0 < t < 8$, the velocity is increasing on the interval $0 < t < 8$, Since the acceleration is negative on the interval $8 < t < 16$ the velocity is decreasing on the interval $8 < t < 16$.
Specific behaviours
✓ references acceleration graph or function ✓ recognises that the upward velocity is increasing on the interval $0 < t < 8$ ✓ recognises that the upward velocity is decreasing on the interval $8 < t < 16$

(c) Suppose that the ground floor has displacement $x = 0$ m. Determine the displacement function of the elevator and hence determine the height above the ground floor of the sixth floor. (3 marks)

Solution
The displacement is the integral of the velocity $$x(t) = -9\cos\left(\frac{\pi t}{16}\right) + c \text{ m}$$ Since $x(0) = 0$ it follows that $$0 = -9\cos(0) + c$$ $$0 = -9 + c$$ $$c = 9$$ Hence $$x(t) = 9 - 9\cos\left(\frac{\pi t}{16}\right) \text{ m}$$ Evaluating $x(16)$ $$x(16) = 9 - 9\cos(\pi)$$ $$= 18 \text{ m}$$
Specific behaviours
✓ integrates the velocity function to obtain the displacement including unknown integration constant ✓ determines integration constant ✓ evaluates $x(16)$

Question 10 **(7 marks)**

A group of researchers conducted a study into the number of siblings of adult Australians citizens. They surveyed a total of 200 participants and recorded the number of siblings, X, for each participant.

A few days later the lead researcher discovered that the survey data had been misplaced. Fortunately, one of the research assistants had been doing some rough calculations on a whiteboard and the lead researcher was able to recover the following information about the probability distribution for X and the mean μ.

x	0	1	2	3
$P(X = x)$	0.2	a	b	0.1

$$\mu = 1.3$$

The letters a and b have been used to denote unknown probabilities.

(a) (i) Write **two** independent equations for a and b. (2 marks)

Solution
Probabilities add to 1. $$0.2 + a + b + 0.1 = 1$$ $$a + b = 0.7$$ Calculation of the mean $$0.2(0) + a(1) + b(2) + 0.1(3) = 1.3$$ $$a + 2b = 1$$
Specific behaviours
✓ recognises that probabilities must add to 1 ✓ implements the formula for the mean

(ii) Hence solve for the unknown probabilities. (2 marks)

Solution
From the first equation $$a = 0.7 - b$$ Substituting into the second equation $$(0.7 - b) + 2b = 1$$ $$b = 0.3$$ $$a = 0.4$$
Specific behaviours
✓ solves for a ✓ solves for b

Later that day the research assistant found the complete probability distribution in their records, and discovered that they had made an error in their original calculation of the mean. The correct probability distribution is given in the table below.

x	0	1	2	3
$P(X=x)$	0.2	0.3	0.4	0.1

(b) (i) Given that there were 200 participants in the study, complete the table below to show the number of participants N with 0, 1, 2 and 3 siblings. (1 mark)

Solution

x	0	1	2	3
$P(X=x)$	0.2	0.3	0.4	0.1
N	40	60	80	20

Specific behaviours

✓ determines correct number of participants in each response group

(ii) Determine the correct mean and standard deviation of the number of siblings X. (2 marks)

Solution

$\mu = 1.4$

$\sigma = 0.9165$

Specific behaviours

✓ determines mean
✓ determines standard deviation

Question 11 **(8 marks)**

A pizza company runs a marketing campaign based on the delivery times of its pizzas. The company claims that it will deliver a pizza in a radius of 5 km within 30 minutes of ordering or it is free. The manager estimates that the actual time, T, from order to delivery is normally distributed with mean 25 minutes and standard deviation 2 minutes.

(a) What it the probability that a pizza is delivered free? (1 mark)

Solution
$P(T > 30) = P\left(Z > \frac{30-25}{2}\right) = P(Z > 2.5) = 1 - 0.9938 = 0.0062$
Specific behaviours
✓ gives the correct value of the probability

(b) On a busy Saturday evening, a total of 50 pizzas are ordered. What is the probability that more than three are delivered free? (2 marks)

Solution
Let Xdenote the number of pizzas out of 50 that are delivered free. Then $X \sim Bin(50,0.0062)$ $P(X > 3) = 1 - P(X \leq 3) = 1 - 0.9997 = 0.0003$
Specific behaviours
✓ states the distribution of the number of pizzas delivered free   ✓ computes the probability correctly

The company wants to reduce the proportion of pizzas that are delivered free to 0.1%.

(c) The manager suggests this can be achieved by increasing the advertised delivery time. What should the advertised delivery time be? (2 marks)

Solution
$P(Z > z) = 0.001 \Rightarrow z = 3.0902 \Rightarrow t = 25 + 3.0902 \times 2 = 31.2$ minutes
Specific behaviours
✓ uses a tail probability of 0.001   ✓ calculates the correct value of the delivery time

After some additional training the company was able to maintain the advertised delivery time as 30 minutes but reduce the proportion of pizzas delivered free to 0.1%.

(d) Assuming that the original mean of 25 minutes is maintained, what is the new standard deviation of delivery times? (3 marks)

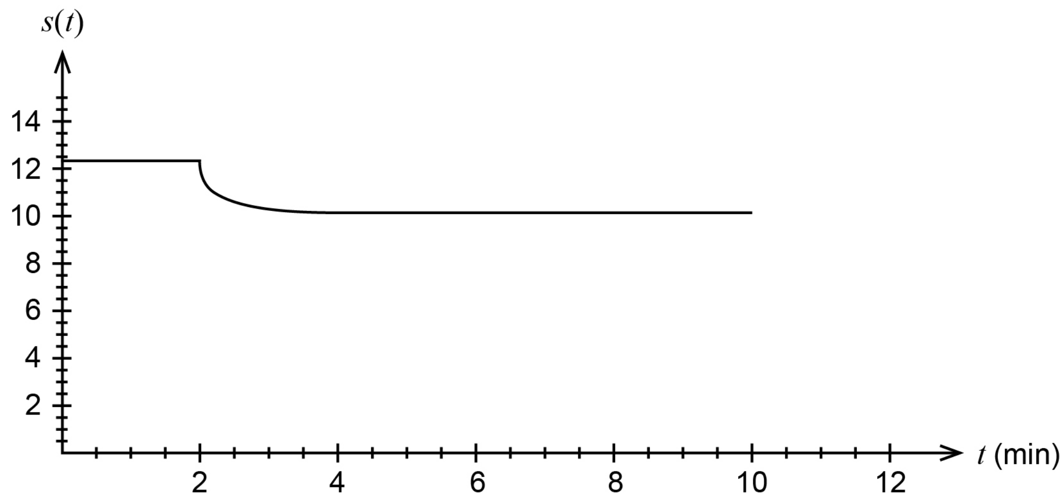
Solution
$z = \frac{30-25}{\sigma} = 3.0902 \Rightarrow \sigma = \frac{30-25}{3.0902} = 1.6$ minutes
Specific behaviours
✓ uses the correct critical value of the normal distribution   ✓ forms the correct equation for σ ✓ solves for σ

Question 12 **(6 marks)**

Part of Josie's workout at her gym involves a 10 minute run on a treadmill. The treadmill's program makes her run at a constant 12.3 km/h for the first 2 minutes and then her speed, $s(t)$, is determined by the equation below, where t is the time in minutes after she began running.

$$s(t) = 10 - \frac{\ln(t - 1.99)}{t} \text{ km/h}$$

(a) Sketch the graph of her speed during this run versus time on the axes below. (3 marks)

Solution

Specific behaviours
✓ correctly graphs $y = 12.3$ ✓ correctly graphs $s(t)$ ✓ shows scale and graphs do not exceed [0,10] domain

(b) At what time(s) is Josie's speed 10 km/h? (1 mark)

Solution
$10 = 10 - \frac{\ln(t - 1.99)}{t}$ $0 = \frac{\ln(t - 1.99)}{t}$ $t = 2.99$ Only point in the given domain is (2.99,10) She runs at 10 km/h when she has run for 2.99 minutes.
Specific behaviours
✓ states the correct time

Question 12 (continued)

(c) At what time(s) during her run is Josie's acceleration zero? (2 marks)

Solution
From CAS calculator: Min at (6.30,9.77) She has zero acceleration for the first 2 minutes of her run and at the instant $t = 6.30$ minutes.
Specific behaviours
✓ states first 2 minutes ✓ states 6.30 minutes

Question 13 **(8 marks)**

The proportion of working adults who miss breakfast on week days is estimated to be 40%. A study takes a random sample of 400 working adults.

(a) For this sample:

(i) What is the (approximate) distribution of the sample proportions of workers who miss breakfast? (2 marks)

Solution
$\hat{p} \sim N\left(0.4, \frac{0.4 \times 0.6}{400}\right)$ That is, $\hat{p} \sim N(0.4, 0.0006)$
Specific behaviours
✓ states normal distribution with correct mean ✓ gives correct value of variance of standard deviation

(ii) What is the probability that the sample proportion of workers who miss breakfast is greater than 44%? (2 marks)

Solution
$P(\hat{p} > 0.44) = P(Z > (0.44 - 0.4)/\sqrt{0.0006}) = P(Z > 1.6330) = 1 - 0.9488$ $= 0.0512$
Specific behaviours
✓ uses distribution from (i) ✓ determines correct probability

Tom takes a random sample of 400 adults. He obtained his sample by selecting the first 400 workers he met in a busy mall in Perth city during lunch time.

(b) Discuss briefly **two** possible sources of bias in Tom's sample. (2 marks)

Solution
1. Location: only one location, so only those present in that mall will be sampled from. 2. Time: lunch time, so only those present at lunch time will be sampled from. 3. Selection scheme: the first 400 workers only are in the sample, so this is not a random sample from all workers.
Specific behaviours
✓ states one source of bias with explanation ✓ states a second source of bias with explanation

Question 13 (continued)

Amir suggests that a better sampling scheme is to obtain a random sample of 400 voters and contact them by telephone.

(c) (i) Outline **one** source of bias in Amir's sampling scheme. (1 mark)

Solution
1. Only those with listed telephone numbers will be selected. 2. Not everyone will answer their phone when called.
Specific behaviours
✓ one source of bias is outlined

(ii) Which of Tom's or Amir's sampling scheme is better? Provide a reason for your choice. (1 mark)

Solution
Amir's scheme is better, as it samples randomly from the whole population of workers.
Specific behaviours
✓ states Amir's is the better sample with reason

Question 14 **(7 marks)**

(a) What is the minimum sample size required to estimate a population proportion to within 0.01 with 95% confidence. (3 marks)

Solution
$$n = \frac{1.96^2 \times 0.5 \times 0.5}{0.01^2} = 9603.6$$ That is, 9604
Specific behaviours
✓ uses correct z-critical value ✓ uses correct formula ✓ gives correct value to nearest integer rounded up

(b) Identify **two** factors that affect the width of a confidence interval for a population proportion and describe the effect of each. (4 marks)

Solution
Any two of: 1. Sample size: as sample size increases the width decreases. 2. Sample proportion: as sample proportion moves away from 0.5 the width decreases. 3. Confidence level: as confidence level increases the width increase.
Specific behaviours
✓ states one correct reason ✓ states the effect ✓ states two correct reasons ✓ states effect

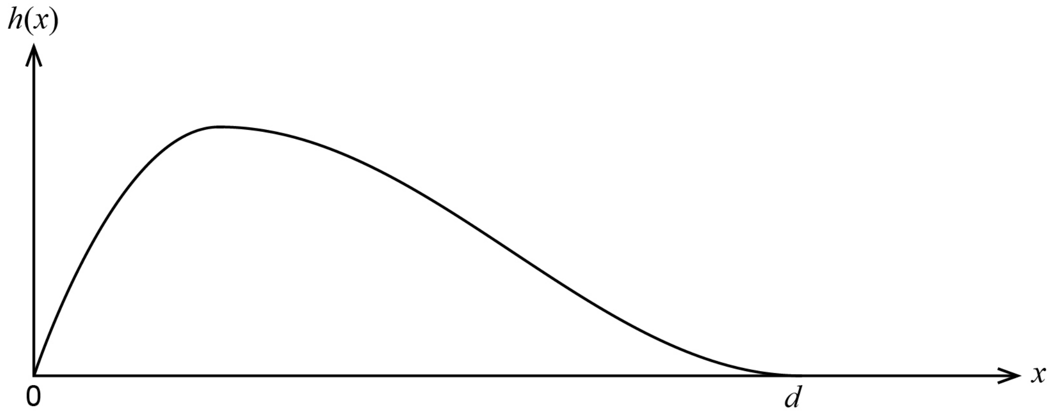
Question 15 **(14 marks)**

A wall in a new Western Australian hotel is to feature a rolling, wave-shaped window. Engineers have modelled the top edge of the wave shape by joining together two functions,

$$h_1(x) = 4 - 4(x-1)^2, \quad 0 \le x \le 1 \text{ and}$$

$$h_2(x) = a(\cos(x-1)+1), \quad 1 < x \le d \quad a, d \text{ constants.}$$

The functions give the height, h, above ground level of the top edge of the window measured in metres. The origin is defined as the leftmost point of the window which is at ground level and x is the horizontal distance to the right of the origin measured in metres. The graph of the two functions is shown below.



(a) Determine the value of the constant a in the function $h_2(x) = a(\cos(x-1)+1)$.

(3 marks)

Solution
Functions meet at $x = 1$. Highest point is $h_1(1) = 4 - 4(1-1)^2$ $= 4$ Functions meet at (1,4) $h_2(1) = 4$ $4 = a(\cos(1-1)+1)$ $4 = 2a$ $a = 2$
Specific behaviours
✓ states highest point is $h_1(1)$ ✓ recognises the need to solve $h_2(1) = h_1(1)$ ✓ determines the value of a

Question 15 (continued)

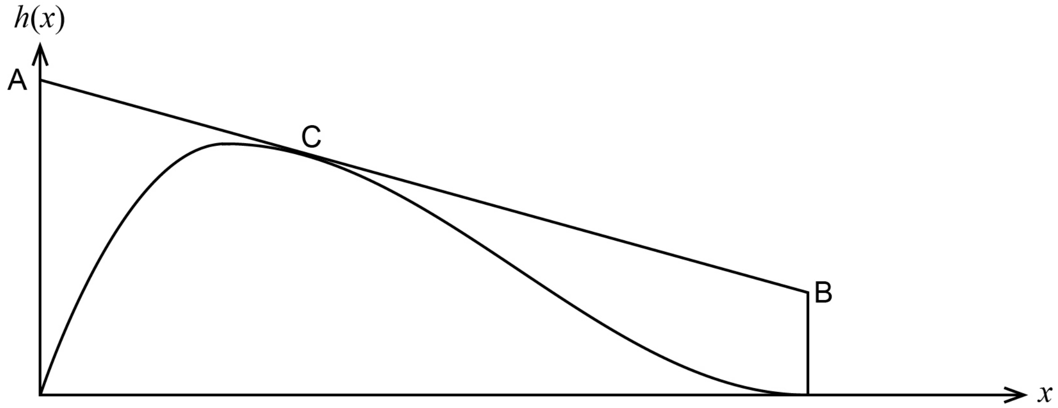
(b) Determine the length of the bottom edge of the window. (2 marks)

Solution
$h_2(d) = 0$ $0 = 2(\cos(d-1)+1)$ $d = \pi + 1$ $= 4.14$ The bottom edge of the window is 4.14 metres long.
Specific behaviours
✓ recognises the need to solve $h_2(d) = 0$ ✓ solves for d

(c) Determine the volume of glass required for the window if it has a uniform thickness of 3 cm. (5 marks)

Solution
$A = \int_0^1 h_1(x)dx + \int_1^{\pi+1} h_2(x)dx$ $\int_0^1 h_1(x)dx = \frac{8}{3}$ $\int_1^{\pi+1} h_2(x)dx = 2\pi$ $A = \frac{8}{3} + 2\pi$ $= 8.95$ $V = \left(\frac{8}{3} + 2\pi\right) \times 0.03$ $= 0.27$ The window has a volume of 0.27 cubic metres
Specific behaviours
✓ recognises the required area is the sum of the areas under $h_1(x)$ and $h_2(x)$ ✓ determines the area under $h_1(x)$ ✓ determines the area under $h_2(x)$ ✓ determines the total area ✓ determines the volume including units

The top edge of the wall, shown as the line AB below, is to just touch the window at the point C shown below. Point A is 1.39 m above the point B.

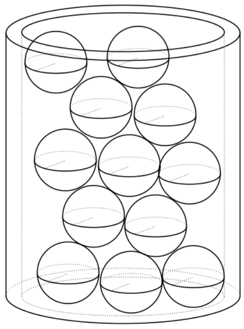


(d) How high is point C above the ground? (4 marks)

Solution
$m = \dfrac{1.39}{0-(\pi+1)}$ $= -0.3356$ $h_2'(x) = -2\sin(x-1)$ $-0.3356 = -2\sin(x-1)$ $x = 1.1686,\ 3.9730$ Solution is 1.1686 $h_2(1.1686) = 3.9716$ C is 3.97 metres above the ground.
Specific behaviours
✓ determines the gradient, m, of the line segment AB ✓ differentiates $h_2(x)$ ✓ equates $h_2'(x)$ to m and solves for x ✓ determines the height of point C

Question 16 **(10 marks)**

A cylindrical glass vase is filled with 20 spherical Christmas decorations as shown below (not all the decorations are visible). All the decorations have a diameter of one-third the internal diameter of the vase and they are completely contained within the vase. For design purposes the sum of the internal diameter of the base of the vase and the vase's internal height is to be 42 cm.



(a) Show that the volume of unused space in the vase, V, can be expressed as a function of the internal radius of the vase, r, and is given below as (3 marks)

$$V(r) = 2\pi\left(21r^2 - \frac{121}{81}r^3\right).$$

Solution
$2r + h = 42$ $h = 42 - 2r$ $V(r,h) = \pi r^2 h - 20\left(\frac{4}{3}\pi\left(\frac{r}{3}\right)^3\right)$ $V(r) = \pi r^2(42-2r) - \frac{80\pi}{81}r^3$ $= 2\pi\left(21r^2 - r^3 - \frac{40}{81}r^3\right)$ $= 2\pi\left(21r^2 - \frac{121}{81}r^3\right)$
Specific behaviours
✓ determines an expression for h in terms of r ✓ states an expression for the volume of unused space in terms of h and r ✓ clearly shows that the expression for h in terms of r can substitute into V and simplifies to determine required result

(b) Use calculus to determine the dimensions of the vase that will maximise the unused space in it. Give your answers rounded to the nearest millimetre. (4 marks)

Solution
$V'(r) = 2\pi\left(42r - \frac{363r^2}{81}\right)$ $0 = 42r - \frac{363r^2}{81}$ $0 = r(42 - \frac{363r}{81})$ $r = 0 \text{ or } \frac{1134}{121}\{= 9.372(3dp)\}$ $V''(r) = 2\pi\left(42 - \frac{726r}{81}\right)$ $V''(9.372) = -ve\{= -84\pi\} \Rightarrow \text{max}$ Dimensions are: $r = 9.4$ cm and $h = 23.3$ cm
Specific behaviours
✓ determines first derivative of $V(r)$ ✓ equates to zero and determines 0 and 9.4 are solutions ✓ clearly shows the use of the second derivative or sign test to show that $r = 9.4$ is a maximum ✓ states the dimensions of the vase that maximise the unused space rounded to the nearest mm

(c) Can more than 20 of the spherical decorations fit inside the vase in part (b)? Use calculations to verify your answer. (3 marks)

Solution
$V(9.4) = 3863.1 \text{ cm}^3$ $V(\text{decoration}) = 127.7 \text{ cm}^3$ There is likely space for more decorations, but it is not certain as it would depend on the way the balls were packed into the vase.
Specific behaviours
✓ states the volume of unused space and the volume of one decoration ✓ infers likely to fit more ✓ states the limitation of packing

Question 17 **(15 marks)**

A microbiologist is studying the effect of temperature on the growth of a certain type of bacteria under controlled laboratory conditions. A population of bacteria is incubated at a temperature of 30 °C and the size of the population measured at hourly intervals for six hours. The logarithm of the population size appears to lie on a straight line when plotted against time (measured in hours) and the line of best fit shown on the axes below.

(a) (i) On the basis of the graph above, what is the size of the bacteria population after two hours? (2 marks)

Solution
$\log_{10}(P) = 3$ $P = 10^3 = 1000$
Specific behaviours
✓ identifies the correct value of $\log_{10}(P)$ ✓ uses the inverse relationship between $\log_{10}(x)$ and 10^x to solve for P

(ii) The equation of the line can be written in the form $\log_{10}(P) = At + B$. Use the graph to determine the values of A and B. (2 marks)

Solution
Gradient of ½ and vertical axis intercept of 2 $\log_{10}(P) = \frac{1}{2}t + 2$
Specific behaviours
✓ determines the correct value for A ✓ determines the correct value for B

Another population of the same bacteria is cultured at 40 °C. The size of the population, P, after t hours satisfies the equation

$$\log_{10}(P) = \frac{1}{3}t + 2.$$

(b) (i) Express the above equation in the form $P = A(10)^{Bt}$. (3 marks)

Solution
$\log_{10}(P) = \frac{1}{3}t + 2$ $P = 10^{t/3+2}$ $P = 10^{t/3}10^2$ $P = 100 \cdot 10^{t/3}$
Specific behaviours
✓ uses the inverse relationship between $\log_{10}(x)$ and 10^x ✓ uses the appropriate index law ✓ determines correct expression

(ii) Determine the size of the population after exactly four hours to the nearest whole number. (1 mark)

Solution
$P = 100 \cdot 10^{4/3}$ $P \approx 2154$
Specific behaviours
✓ evaluates the population rounded to a whole number

(iii) Express the above equation in the form $t = C\log_{10}\left(\frac{P}{D}\right)$. (3 marks)

Solution
$\log_{10}(P) = \frac{1}{3}t + 2$ $\log_{10}(P) - 2 = \frac{1}{3}t$ $\log_{10}(P) - \log_{10}(100) = \frac{1}{3}t$ $\log_{10}\left(\frac{P}{100}\right) = \frac{1}{3}t$ $t = 3\log_{10}\left(\frac{P}{100}\right)$
Specific behaviours
✓ expresses 2 in terms of a log of base 10 ✓ applies appropriate log law to arrive at single log expression (second last line) ✓ determines correct expression

Question 17 (continued)

(iv) How many minutes does it take for the population to reach a size of 5000? Give your answer to the nearest minute. (2 marks)

Solution
$t = 3\log_{10}\left(\frac{5000}{100}\right)$ $t \approx 5.0969 \text{ hours}$ $t \approx 306\ minutes$
Specific behaviours
✓ evaluates the time in hours ✓ converts to minutes (rounded to the nearest minute)

(c) With reference to parts (a) and (b), describe the effect of temperature on the population growth of this type of bacteria. (2 marks)

Solution
The equation at 30 degrees has a greater slope than that of the 40 degree equation which indicates a greater growth rate. Parts (a) and (b) would seem to indicate that the lower temperature incubation results in a higher growth rate.
Specific behaviours
✓ identifies features of the equations in parts (a) and (b) that relate to growth ✓ states lower temperature has higher growth

Question 18 **(9 marks)**

A building has five alarms configured in such a way that the system functions if at least two of the alarms work. The probability that an alarm fails overnight is 0.05. Let the random variable X denote the number of alarms that fail overnight.

(a) State the distribution of X. (2 marks)

Solution
$X \sim Bin(5,0.05)$
Specific behaviours
✓ states the binomial distribution ✓ gives the correct parameters

(b) What assumptions are required for the distribution in part (a) to be valid? (2 marks)

Solution
1. The alarms fail independent of each other. 2. The probability that an alarm fails is constant/unchanging/same for all alarms.
Specific behaviours
✓ states one correct assumption ✓ states the second correct assumption

(c) What is the probability that the alarm system fails overnight? (2 marks)

Solution
We need $P(X \geq 4) = 1 - P(X \leq 3) = 1 - 0.99997 = 0.00003$
Specific behaviours
✓ writes the first probability statement correctly ✓ obtains the correct final answer to at least 5 decimal places

One of the alarms is removed in the evening for maintenance and is not replaced.

(d) What is the probability that the alarm system still works in the morning? (3 marks)

Solution
Let the random variable Y denote the number of alarms that fail out of 4. Then $Y \sim Bin(4,0.05)$. We need $P(Y \geq 3) = 0.00048$ $1 - 0.00048 = 0.99952$
Specific behaviours
✓ states the distribution of the random variable with correct parameters ✓ writes the first probability statement correctly ✓ obtains the correct final answer

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