Government of **Western Australia**
**School Curriculum and Standards Authority**

# MATHEMATICS SPECIALIST

## Calculator-free

## ATAR course examination 2021

## Marking key

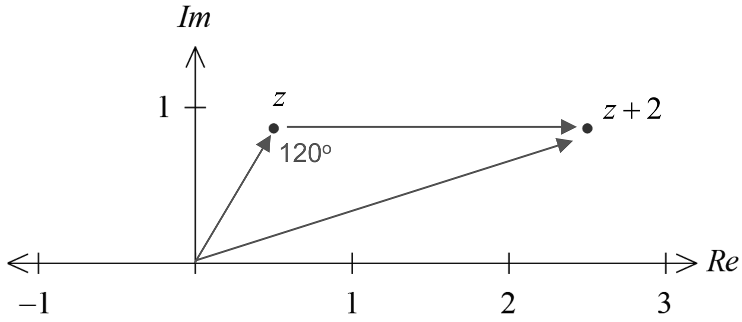
Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2021/67372

**Section One: Calculator-free** **35% (49 Marks)**

**Question 1** **(4 marks)**

The Argand diagram below shows the complex numbers $z$ and $z+2$ where $z = cis\left(\frac{\pi}{3}\right)$.




Determine the exact value for:

(a) $Arg(-z)$. (1 mark)

| Solution |
|---|
| $Arg(-z) = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$. <br> Also accept $Arg(-z) = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$ |
| **Specific behaviours** |
| ✓ states the correct value |

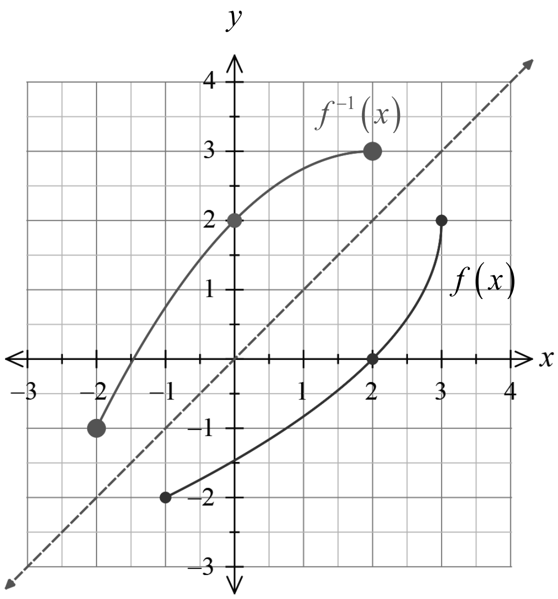
(b) $\lvert z+2 \rvert$. (3 marks)

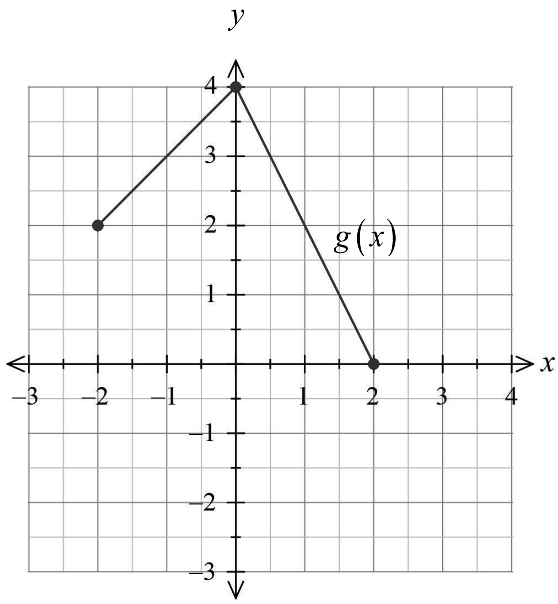
| Solution |
|---|
| Applying the cosine rule $\lvert z+2\rvert^2 = 1^2 + 2^2 - 2(1)(2)\cos\left(\frac{2\pi}{3}\right)$ <br> $= 1+4-4\left(-\frac{1}{2}\right) = 5+2 = 7$ <br> $\therefore \lvert z+2\rvert = \sqrt{7}$ |
| **Specific behaviours** |
| ✓ determines an angle of 120° between the vectors representing $z$ and $z+2$ <br> ✓ applies the cosine rule correctly <br> ✓ determines the value for $\lvert z+2\rvert$ correctly |

| Alternative Solution |
|---|
| $z = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \quad \therefore \quad z+2 = \left(\frac{5}{2}\right) + \frac{\sqrt{3}}{2}i$ <br> $\lvert z+2\rvert^2 = \left(\frac{5}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{25}{4} + \frac{3}{4} = 7$ <br> $\therefore \quad \lvert z+2\rvert = \sqrt{7}$ |
| **Specific behaviours** |
| ✓ determines $z+2$ in Cartesian form correctly <br> ✓ forms the expression for $\lvert z+2\rvert^2$ correctly <br> ✓ determines the value for $\lvert z+2\rvert$ correctly |

**Question 2** **(11 marks)**

The graphs of functions $f$ and $g$ are shown below.




(a) Sketch the graph of function $f^{-1}$ on the same axes used for function $f$. (2 marks)

| Solution |
|---|
| See above graph axes. |
| **Specific behaviours** |
| ✓ indicates a concave down curve that is a reflection of $y=f(x)$ about $y=x$ <br> ✓ indicates all the points $(-2,-1)$, $(0,2)$ and $(2,3)$ |

(b) Explain why the inverse of $g$ is not a function. (1 mark)

| Solution |
|---|
| Function $g$ is not a one-to-one function over its domain OR does not pass the horizontal line test. |
| **Specific behaviours** |
| ✓ refers to function $g$ not being a one-to-one function |

The defining rule for function $f$ is $f(x) = 2-2\sqrt{3-x}$ where $-1 \le x \le 3$.

(c) Determine the rule for $y = f^{-1}(x)$. (3 marks)

| Solution |
|---|
| $f:\ y = 2-2\sqrt{3-x}$ Hence $f^{-1}:\ x = 2-2\sqrt{3-y}$ |
| $\sqrt{3-y} = \dfrac{2-x}{2}$ $\therefore\ 3-y = \left(\dfrac{2-x}{2}\right)^2$ $\therefore\ f^{-1}(x) = 3 - \left(\dfrac{2-x}{2}\right)^2$ |
| **Specific behaviours** |
| ✓ interchanges $x, y$ to obtain the rule for the inverse<br>✓ obtains the correct expression for $\sqrt{3-y}$<br>✓ obtains the correct defining rule for $y = f^{-1}(x)$ |

(d) Determine the exact value for $g\left(f(0)\right)$. (2 marks)

| Solution |
|---|
| $g\left(f(0)\right) = g\left(2-2\sqrt{3}\right)$<br>$= \left(2-2\sqrt{3}\right) + 4$ since $-2 \le 2-2\sqrt{3} \le 0$<br>$= 6-2\sqrt{3}$ |
| **Specific behaviours** |
| ✓ evaluates $f(0)$ correctly<br>✓ determines the exact value $6-2\sqrt{3}$ correctly |

(e) Determine the domain for the function $y = f\left(g(x)\right)$. Justify your answer. (3 marks)

| Solution |
|---|
| The range of $g$ must be a SUBSET of the domain of $f$.<br>$\therefore\ D_{fog} = \{\, x \mid -2 \le x \le -1,\ 0.5 \le x \le 2 \}$<br><br>Note that for $-1 < x < 0.5$ $g(x) > 3$ which is not in the domain for function $f$. |
| **Specific behaviours** |
| ✓ states that $-2 \le x \le -1$<br>✓ states that $0.5 \le x \le 2$<br>✓ justifies the chosen domain correctly |

**Question 3** **(5 marks)**

Using an appropriate substitution, determine the exact value for $\int_2^3 15x\sqrt{x-2}\,dx$.

**Solution**

Using $u = x - 2$

| $x$ | 2 | 3 |
|---|---|---|
| $u$ | 0 | 1 |

$$\frac{du}{dx} = 1 \quad \therefore \quad dx = du$$

$$\int_2^3 15x\sqrt{x-2}\,dx = \int_0^1 15(u+2)\left(u^{\frac{1}{2}}\right).\,du$$

$$= 15\int_0^1 \left(u^{\frac{3}{2}} + 2u^{\frac{1}{2}}\right)du$$

$$= 15\left[\frac{2u^{\frac{5}{2}}}{5} + \frac{4u^{\frac{3}{2}}}{3}\right]_0^1 = 15\left[\left(\frac{2}{5}+\frac{4}{3}\right)-(0+0)\right] = 26$$

**Specific behaviours**

- ✓ changes the limits correctly for the chosen substitution
- ✓ obtains $dx$ in terms of $du$ correctly
- ✓ simplifies the integrand correctly using the chosen substitution
- ✓ anti-differentiates correctly
- ✓ evaluates the definite integral correctly

**Alternative Solution**

Using $u = \sqrt{x-2}$

| $x$ | 2 | 3 |
|---|---|---|
| $u$ | 0 | 1 |

$$\frac{du}{dx} = \frac{1}{2\sqrt{x-2}} \quad \therefore \quad dx = 2\sqrt{x-2}\,du = 2u\,du$$

$$\int_2^3 15x\sqrt{x-2}\,dx = \int_0^1 15\left(u^2+2\right)(u).\,2u\,du$$

$$= 15\int_0^1 \left(2u^4 + 4u^2\right)du$$

$$= 15\left[\frac{2u^5}{5} + \frac{4u^3}{3}\right]_0^1 = 15\left[\left(\frac{2}{5}+\frac{4}{3}\right)-(0+0)\right] = 26$$

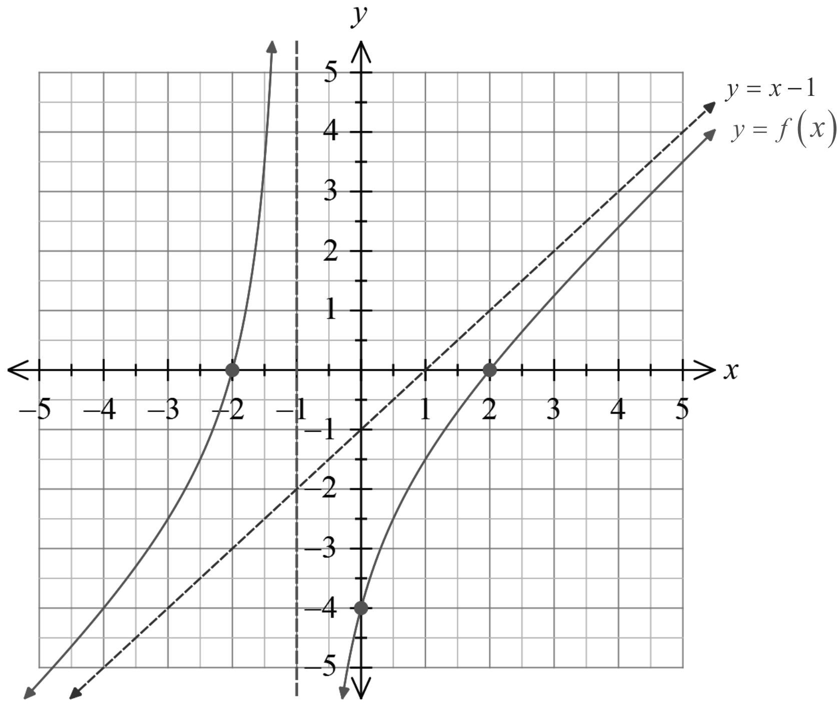
**Specific behaviours**

- ✓ changes the limits correctly for the chosen substitution
- ✓ obtains $dx$ in terms of $du$ correctly
- ✓ simplifies the integrand correctly using the chosen substitution
- ✓ anti-differentiates correctly
- ✓ evaluates the definite integral correctly

**Question 4** **(5 marks)**

Consider the function $f(x) = \dfrac{x^2-4}{x+1} = x-1-\dfrac{3}{x+1}$.

Sketch the graph of the function $y=f(x)$ on the axes below. Indicate clearly the $x$ and $y$ intercepts and any asymptotes.




| Solution |
|---|
| $f(x) = \dfrac{(x+2)(x-2)}{(x+1)} = x-1-\dfrac{3}{x+1}$ |
| $x$ intercepts occur when $x^2-4=0$ *i.e.* at $x=\pm 2$ $y$ intercept $f(0)=-4$ |
| Vertical asymptote is $x=-1$. |
| As $\lvert x\rvert \to \infty$, $f(x)\to x-1$ (inclined asymptote) |
| Sketch shown above. |
| **Specific behaviours** |
| ✓ indicates $x$ intercepts at $x=\pm 2$ |
| ✓ indicates a vertical asymptote at $x=-1$ |
| ✓ indicates $f(0)=-4$ |
| ✓ indicates inclined asymptote $y=x-1$ i.e. $f(x)\to x-1$ for $\lvert x\rvert\to\infty$ |
| ✓ indicates correct curvature in the graph |

**Question 5** **(5 marks)**

(a) Given that $\dfrac{7x^2-12x+2}{(x-2)(x^2+2)} = \dfrac{a}{x-2} + \dfrac{bx}{x^2+2}$ determine the values of $a$ and $b$.

(2 marks)

| **Solution** |
|---|
| $\dfrac{a}{x-2} + \dfrac{bx}{x^2+2} = \dfrac{a(x^2+2)+bx(x-2)}{(x-2)(x^2+2)} = \dfrac{(a+b)x^2-2bx+2a}{(x-2)(x^2+2)}$<br>Equating coefficients: $a+b=7$<br>$-2b=-12$<br>$2a=2$<br>Solving gives $a=1,\quad b=6$ |
| **Specific behaviours** |
| ✓ forms the equivalence of numerators correctly<br>✓ solves for $a,b$ correctly |

(b) Hence determine $\displaystyle\int \frac{7x^2-12x+2}{(x-2)(x^2+2)}\,dx$ . (3 marks)

| **Solution** |
|---|
| $\displaystyle\int \frac{7x^2-12x+2}{(x-2)(x^2+2)}\,dx = \int\left(\frac{1}{x-2}+\frac{6x}{x^2+2}\right)dx$<br>$\displaystyle = \int\frac{1}{x-2}\,dx + 3\int\frac{2x}{x^2+2}\,dx$<br>$= \ln\lvert x-2\rvert + 3\ln\left(x^2+2\right) + k$ |
| **Specific behaviours** |
| ✓ re-writes the integrand correctly in terms of the partial fractions<br>✓ anti-differentiates the $(x-2)^{-1}$ term correctly using the absolute value of a natural logarithm AND uses an integration constant<br>✓ anti-differentiates the $6x(x^2+2)^{-1}$ term correctly (absolute value not required) |

**Question 6** **(5 marks)**

Consider the quartic polynomial $P(z) = z^4 - 6z^3 + 31z^2 - 52z + 60$.

(a) Given that $P(2+4i) = 0$, determine a quadratic factor of $P(z)$. (2 marks)

| Solution |
|---|
| Since $P(2+4i)=0$ then we also have $P(2-4i)=0$ as all coefficients are real.<br>$Q(z) = (z-(2+4i))(z-(2-4i))$<br>$= (z^2 - 4z + 20)$ |
| **Specific behaviours** |
| ✓ states that $P(2-4i)=0$ or states that $z-(2-4i)$ is a factor<br>✓ determines the quadratic factor $Q(z)$ correctly |

(b) Hence solve the equation $z^4 - 6z^3 + 31z^2 - 52z + 60 = 0$. (3 marks)

| Solution |
|---|
| $P(z) = (z^2 - 4z + 20)(z^2 - 2z + 3)$ i.e. $T(z) = z^2 - 2z + 3$<br>i.e. $P(z) = (z^2 - 4z + 20)((z-1)^2 + 2)$<br>$= (z-(2+4i))(z-(2-4i))(z-(1+\sqrt{2}i))(z-(1-\sqrt{2}i))$<br><br>Solving $T(z)=0$ gives $z = 1 \pm \sqrt{2}\,i$<br>Solutions are $z = 2 \pm 4i,\ 1 \pm \sqrt{2}\,i$ |
| **Specific behaviours** |
| ✓ determines the quadratic factor $T(z)$ correctly<br>✓ states that $z = 1 + \sqrt{2}\,i$ is a solution<br>✓ states that $z = 1 - \sqrt{2}\,i$ is a solution |

**Question 7** **(5 marks)**

The number $2021$ can be expressed as a product of two consecutive prime numbers: $43 \times 47 = 2021$.

Consider the complex equation $z^{43} = 1$.

(a) Write an expression for the roots of $z^{43} = 1$. (2 marks)

| **Solution** |
|---|
| The equation $z^{43} = 1$ has 43 roots where any root is of the form given by: $w = cis\left(\frac{2\pi k}{43}\right)$ where $k = 0, 1, 2, \ \ldots\ , 42.$ |
| **Specific behaviours** |
| ✓ writes the correct form $cis\left(\frac{2\pi k}{43}\right)$<br>✓ states that the integer $k = 0, 1, 2, \ \ldots\ , 42.$ |

Let $w$ be any one of the roots of the equation $z^{43} = 1$.

(b) How many of these roots will also be a solution of the equation $z^{47} = 1$? Justify your answer. (3 marks)

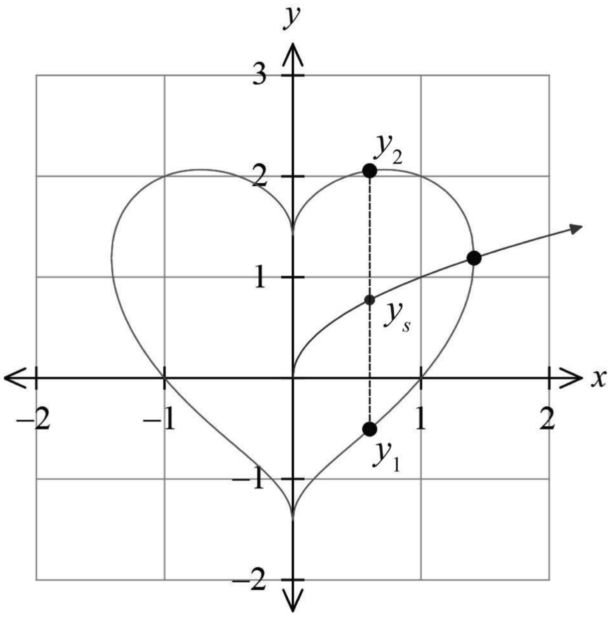
| **Solution** |
|---|
| If $w$ is also a root of $z^{47} = 1$ then we must show that $w^{47} = 1$.<br>Examining the expression $w^{47} = \left(cis\left(\frac{2\pi k}{43}\right)\right)^{47} \qquad k = 0, 1, 2, \ldots, 42$<br>$= cis\left(\frac{47 \times 2\pi k}{43}\right) = cis\left(\frac{47 \times k \times 2\pi}{43}\right)$<br>This will be equal to ONE if and only if $\frac{47 \times k}{43}$ is an integer. If this occurs then the argument for $w^{47}$ will be a multiple of $2\pi$ and hence $w^{47} = 1$.<br>Since 43 and 47 are both prime numbers, then 43 does not divide into 47 and that 43 will not divide into $k$ when $k = 1, 2, \ \ldots\ , 42$ .<br>Hence $\frac{47 \times k}{43}$ can never be an integer where $k = 1, 2, \ \ldots\ , 42$.<br>When $k = 0$, $w = 1$ is a solution of BOTH $z^{43} = 1$ and $z^{47} = 1$.<br>$\therefore$ *Only ONE* of the roots $(w = 1)$ of $z^{43} = 1$ is also a root of $z^{47} = 1$. |
| **Specific behaviours** |
| ✓ forms the expression for $w^{47}$ correctly in terms of the integer $k$<br>✓ states that only ONE of the roots $(w = 1)$ of $z^{43} = 1$ is also a root of $z^{47} = 1$<br>✓ justifies the answer using the fact that the argument for $w^{47}$ is never an even multiple of $\pi$ (for $k \neq 0$) |

| Alternative Solution |
|---|
| The equation $z^{43} = 1$ has 43 roots where any root is of the form given by:<br>$w = cis\left(\frac{2\pi k}{43}\right)$ where $k = 0,1,2,\ ...\ ,42.$<br>If $w$ is also a root of $z^{47} = 1$ then $w = cis\left(\frac{2\pi m}{47}\right)$ where $m = 0,1,2,\ ...\ ,46.$<br>Hence we require : $w = cis\left(\frac{2\pi k}{43}\right) = cis\left(\frac{2\pi m}{47}\right)$<br>Hence $\left(\frac{2\pi k}{43}\right) = \left(\frac{2\pi m}{47}\right)$ where $k = 0,1,2,\ ...\ ,42$ and $m = 0,1,2,\ ...\ ,46.$<br>i.e. $\left(\frac{k}{43}\right) = \left(\frac{m}{47}\right)$<br>i.e. $m = \frac{47 \times k}{43}$ must be an integer.<br><br>Since 43 and 47 are both prime numbers, then 43 does not divide into 47 and that 43 will not divide into $k$ when $k = 1,2,\ ...\ ,42$ .<br>Hence $\frac{47 \times k}{43}$ can never be an integer where $k = 1,2,\ ...\ ,42$.<br><br>When $k = 0,\ m = 0$, then $w = 1$ is a solution of BOTH $z^{43} = 1$ and $z^{47} = 1$.<br><br>$\therefore$ *Only ONE* of the roots $(w = 1)$ of $z^{43} = 1$ is also a root of $z^{47} = 1$. |
| **Specific behaviours** |
| ✓ forms the expression for the roots of $z^{47} = 1$ correctly in terms of the integer $m$ (a different parameter to $k$ )<br>✓ states that only ONE of the roots $(w = 1)$ of $z^{43} = 1$ is also a root of $z^{47} = 1$<br>✓ justifies the answer using the fact $m = \frac{47 \times k}{43}$ cannot be an integer (unless both $k = 0,\ m = 0$) |

**Question 8** **(9 marks)**

The heart-shaped figure shown is given by the equation $x^2 + \left(y - \sqrt{|x|}\right)^2 = 2$.

For $x \geq 0,$ this equation becomes $x^2 + \left(y - \sqrt{x}\right)^2 = 2$. The curve $y = \sqrt{x}$ is also drawn.
This heart-shaped curve has the special property that for each $x$ coordinate in its domain its two $y$ coordinates are an equal vertical distance from the curve $y = \sqrt{x}$.




(a) Explain why the domain for the curve given by $x^2 + \left(y - \sqrt{x}\right)^2 = 2$ is $0 \leq x \leq \sqrt{2}$.

(2 marks)

| **Solution** |
|---|
| From $x^2 + \left(y - \sqrt{x}\right)^2 = 2$ $\qquad \left(y - \sqrt{x}\right)^2 = 2 - x^2$ <br> $\therefore \; y - \sqrt{x} = \pm\sqrt{2 - x^2}$ <br> *i.e.* $y = \sqrt{x} \pm \sqrt{2 - x^2}$ <br> Hence for $\sqrt{2 - x^2}$ and $\sqrt{x}$ to exist we require $2 - x^2 \geq 0$ and $x \geq 0$ <br> i.e. $x^2 \leq 2 \qquad \therefore \quad 0 \leq x \leq \sqrt{2}$ |
| **Specific behaviours** |
| ✓ states that $\sqrt{x}$ must exist <br> ✓ states that $\sqrt{2 - x^2}$ must exist or states that $2 - x^2 \geq 0$ |

| **Alternative Solution** |
|---|
| Intersection of $x^2 + \left(y - \sqrt{x}\right)^2 = 2$ and $y = \sqrt{x}$ is given by : <br> $x^2 + (0)^2 = 2 \quad$ *i.e.* $x^2 = 2 \quad \therefore \; x = \sqrt{2}$ Hence from graph $0 \leq x \leq \sqrt{2}$ |
| **Specific behaviours** |
| ✓ uses the idea of the intersection of $x^2 + \left(y - \sqrt{x}\right)^2 = 2$ and $y = \sqrt{x}$ <br> ✓ obtains solution $x = \sqrt{2}$ |

(b) Show that the total area enclosed by the heart-shaped figure is given by:

$Area = 4\int_0^{\sqrt{2}} \sqrt{2-x^2}\, dx$. (2 marks)

| Solution |
|---|
| From $x^2 + \left(y-\sqrt{x}\right)^2 = 2 \qquad \left(y-\sqrt{x}\right)^2 = 2-x^2$ <br> $i.e.\quad y_2 - y_s = \sqrt{2-x^2}$ <br> $y_2 - y_1 = 2\left(y_2 - y_s\right) = 2\sqrt{2-x^2}$ <br> Area $= \int_{-\sqrt{2}}^{\sqrt{2}} \left(y_2 - y_1\right) dx = 2\int_0^{\sqrt{2}} \left(y_2 - y_1\right) dx$ (symmetry about $x=0$) <br> $= 2\int_0^{\sqrt{2}} \left(2\left(y_2 - y_s\right)\right) dx$ <br> $= 2\int_0^{\sqrt{2}} 2\sqrt{2-x^2}\, dx = 4\int_0^{\sqrt{2}} \sqrt{2-x^2}\, dx$ |
| **Specific behaviours** |
| ✓ indicates symmetry about $x=0$ to obtain one factor of 2 <br> ✓ obtains the integrand $2\sqrt{2-x^2}$ from the two curves |

**Question 8** (continued)

(c) By using the substitution $x = \sqrt{2}\sin\theta$, evaluate the total area enclosed by the heart-shaped figure, and hence see why it can be said that '$\pi$ is at the heart of mathematics'. (5 marks)

**Solution**

Using $x = \sqrt{2}\sin\theta$

| $x$ | $0$ | $\sqrt{2}$ |
|---|---|---|
| $u$ | $0$ | $\frac{\pi}{2}$ |

$$\frac{dx}{d\theta} = \sqrt{2}\cos\theta \quad \therefore \quad dx = \sqrt{2}\cos\theta\, d\theta$$

$$Area = 4\int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx = 4\int_0^{\frac{\pi}{2}} \sqrt{2-2\sin^2\theta}\,.\sqrt{2}\cos\theta\,.\,d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{2\cos^2\theta}\,.\,\sqrt{2}\cos\theta\,.\,d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} 2\cos^2\theta\,d\theta = 4\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta = 4\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 4\left[\left(\frac{\pi}{2}+0\right)-(0+0)\right]$$

$\therefore \quad Area = 2\pi$ square units

**Specific behaviours**

- ✓ changes the limits correctly
- ✓ obtains $dx$ in terms of $d\theta$ correctly
- ✓ uses the Pythagorean identity and the cosine double angle identity to simplify the integrand in terms of $\theta$
- ✓ anti-differentiates correctly
- ✓ evaluates the definite integral correctly to obtain the correct area

An *Acknowledgements variation* document is available on the Authority website.

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