Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS SPECIALIST

Calculator-assumed

ATAR course examination 2022

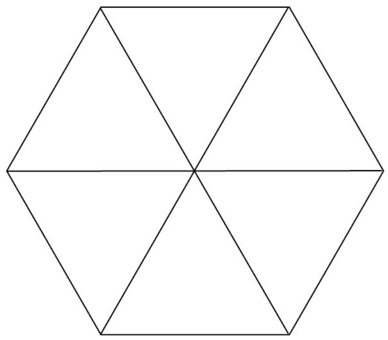
Marking key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

Section Two: Calculator-assumed **65% (86 Marks)**

Question 8 **(4 marks)**

A regular hexagon expands so that the length of each side increases at a rate of 0.5 cm per second. Assuming that the polygon maintains its shape, determine the rate at which the area is increasing when the side length is 4 cm.

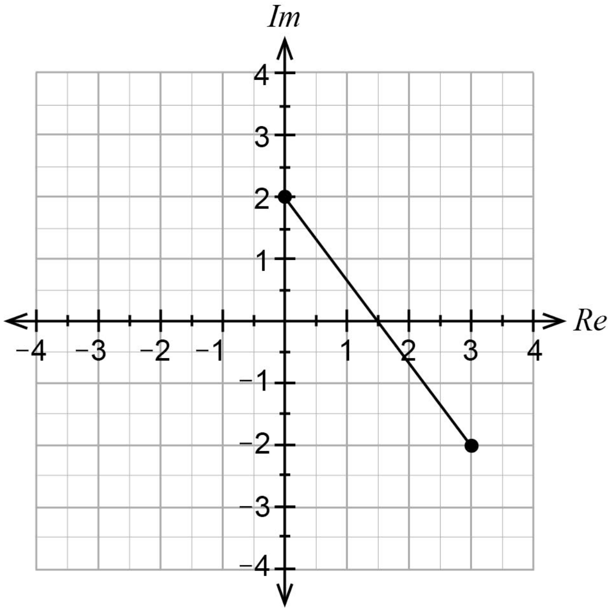


Solution
Let $x =$ side length of the regular hexagon $A = 6 \times Area\left(Equilateral\ \Delta\right)$ $= 6\times\left(\frac{1}{2}\right)\times(x)(x)\left(\sin 60^o\right) = 3x^2.\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}x^2}{2}$ $\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$ $= \left(3\sqrt{3}x\right) \times \frac{dx}{dt}$ $= \left(3\sqrt{3}(4)\right)\times(0.5) = 10.3923...\ cm^2/\sec$ i.e. the area is increasing at a rate of 10.39 cm^2 per second
Specific behaviours
✓ correctly formulates the area in terms of the side length ✓ differentiates correctly to relate the rates of change ✓ substitutes $x = 4$ and $\frac{dx}{dt} = 0.5$ correctly ✓ calculates correctly giving the correct units

Question 9 **(8 marks)**

(a) Sketch the locus of a complex number z satisfying the condition:

$|z-2i| + |z-(3-2i)| = 5$ (2 marks)



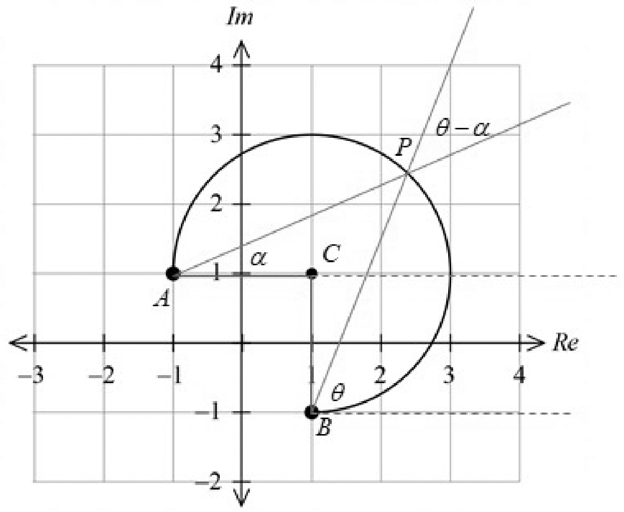
Solution
Locus can be interpreted as: "the distance of z from $2i$ added to the distance of z from $3-2i$ is equal to 5 units." Since the distance from $(0,2)$ and $(3,-2)$ is 5 units, then the locus is a line segment of the points connecting the end points $z=2i$ and $z=3-2i$.
Specific behaviours
✓ indicates a locus that contains $(0,2)$ and $(3,-2)$ ✓ indicates a line segment with correct end points

(b) Describe the locus of the equation $(z+i)(\overline{z+i}) = 2$. (3 marks)

Solution
$(z+i)(\overline{z+i}) = 2$ i.e. $\lvert z+i\rvert^2 = 2$ OR $x^2+(y+1)^2 = 2$ where $x=\mathrm{Re}(z),\ y=\mathrm{Im}(z)$ i.e. $\lvert z-(-i)\rvert = \sqrt{2}$ Description: Locus is a CIRCLE with radius $\sqrt{2}$ and centre $z=-i$.
Specific behaviours
✓ uses appropriate complex number properties to re-write the equation correctly ✓ states that the locus is a CIRCLE with a radius of $\sqrt{2}$ units ✓ states that the centre is $z=-i$ or $(0,-1)$

Question 9 (continued)

(c) The sketch of the locus of a complex number z has been shown below. Write equations or inequalities in terms of z $\left(\text{without using } x = Re(z) \text{ or } y = Im(z)\right)$ for the indicated locus. (3 marks)



Solution

Locus is given by $\left|z-(1+i)\right| = 2$ and $-\frac{\pi}{2} \leq Arg\left(z-(1+i)\right) \leq \pi$

Specific behaviours

✓ states the equation $\left|z-(1+i)\right| = 2$

✓ states an inequality about the argument from $1+i$

✓ states the correct limits for the argument from $1+i$

Alternative Solution 1

Locus is given by $\left|z-(1+i)\right| = 2$ and $-\frac{\pi}{4} \leq Arg(z) \leq \frac{3\pi}{4}$

Specific behaviours

✓ states the equation $\left|z-(1+i)\right| = 2$

✓ states an inequality about the argument from the origin

✓ states the correct limits for the argument from the origin

Alternative Solution 2

Locus is given by $\left|z-(1+i)\right| = 2$

Using the Central Angle Theorem: $s\angle ACB = \frac{\pi}{2} \quad \therefore \; s\angle APB = \frac{\pi}{4}$

$i.e. \quad \theta - \alpha = \frac{\pi}{4} \qquad \therefore \; Arg\left(z-(1-i)\right) - Arg\left(z-(-1+i)\right) = \frac{\pi}{4}$

$$\therefore \quad Arg\left(\frac{z-(1-i)}{z-(-1+i)}\right) = \frac{\pi}{4}$$

Specific behaviours

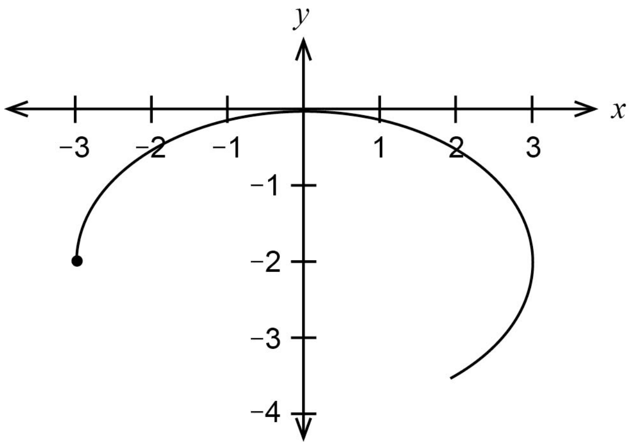
✓ states the equation $\left|z-(1+i)\right| = 2$

✓ writes a difference of arguments equal to $\frac{\pi}{4}$ OR its equivalent

✓ writes the correct expression for each of the two arguments

Question 10 **(9 marks)**

The velocity of a particle is given by $\underset{\sim}{v}(t) = \begin{pmatrix} 3\sin t \\ 2\cos t \end{pmatrix}$ where $t \geq 0$. The particle's initial position vector $\underset{\sim}{r}(0) = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$. The path of the particle is shown for the first 4 seconds.



(a) State what the following definite integrals measure about the motion of the particle:

(i) $\int_0^1 \underset{\sim}{v}(t)\, dt$ (2 marks)

Solution	
$\int_0^1 \underset{\sim}{v}(t)\, dt = \int_0^1 \frac{d}{dt}\left(\underset{\sim}{r}(t)\right) dt = \underset{\sim}{r}(1) - \underset{\sim}{r}(0) = \Delta \underset{\sim}{r}$	This integral gives the change in displacement vector during the first second of motion.
Specific behaviours	
✓ states the integral measures the change in displacement (vector) ✓ states this occurs during the first second of motion	

(ii) $\int_0^{2\pi} \left|\underset{\sim}{v}(t)\right| dt$ (2 marks)

Solution	
$\int_0^{2\pi} \left|\underset{\sim}{v}(t)\right| dt = \int_0^{2\pi} Speed(t)\; dt$	This integral gives the distance travelled for the time it takes to complete one circuit of the path of the particle i.e. the perimeter of the ellipse.
Specific behaviours	
✓ refers to the time it takes for the particle to complete one circuit of its path ✓ states the integral measures the distance travelled	

Question 10 (continued)

(b) Determine $\underset{\sim}{r}(t)$. (3 marks)

Solution
$\underset{\sim}{r}(t) = \int \begin{pmatrix} 3\sin t \\ 2\cos t \end{pmatrix} dt = \begin{pmatrix} -3\cos t \\ 2\sin t \end{pmatrix} + \underset{\sim}{c}$
Using $\underset{\sim}{r}(0) = \begin{pmatrix} -3 \\ -2 \end{pmatrix} \quad \therefore \quad \begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} -3\cos 0 \\ 2\sin 0 \end{pmatrix} + \underset{\sim}{c}$
Hence $\underset{\sim}{c} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$ i.e. $\underset{\sim}{r}(t) = \begin{pmatrix} -3\cos t \\ 2\sin t - 2 \end{pmatrix}$
Specific behaviours
✓ anti-differentiates the velocity vector function correctly using a constant vector ✓ determines the vector constant of integration correctly ✓ uses correct vector and mathematics notation

(c) Determine the Cartesian equation for the path of the particle. (2 marks)

Solution
$\underset{\sim}{r}(t) = \begin{pmatrix} -3\cos t \\ 2\sin t - 2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \therefore \quad \begin{matrix} x = -3\cos t \\ y = 2\sin t - 2 \end{matrix}$
$\therefore \cos t = -\dfrac{x}{3}$ and $\sin t = \dfrac{y+2}{2}$ Using $\cos^2 t + \sin^2 t = 1$ then
$\left(\dfrac{x}{3}\right)^2 + \left(\dfrac{y+2}{2}\right)^2 = 1$ i.e. $\dfrac{x^2}{9} + \dfrac{(y+2)^2}{4} = 1$ (Equation of an ellipse)
Specific behaviours
✓ forms separate expressions for $\cos t$ and $\sin t$ correctly ✓ uses the Pythagorean identity to eliminate t to form a Cartesian equation

Question 11 **(8 marks)**

A Formula One (F1) racing car has an initial displacement of 192 metres with an initial velocity of 24 metres per second. It accelerates for a period of 11 seconds in a straight line so that its velocity v metres per second and displacement x metres are related by the equation:

$$v(x) = \frac{x}{8}$$

(a) Determine the acceleration a as a function of displacement x i.e. determine $a(x)$. (2 marks)

Solution
Acceleration $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(\frac{1}{2}\left(\frac{x}{8}\right)^2\right) = \frac{d}{dx}\left(\frac{x^2}{128}\right) = \frac{x}{64}$ OR $a = v.\frac{dv}{dx} = \left(\frac{x}{8}\right).\frac{1}{8} = \frac{x}{64}$ i.e. $a(x) = \frac{x}{64}$
Specific behaviours
✓ forms a correct expression for $a(x)$ ✓ uses the expression correctly to obtain a in terms of x

(b) Determine the displacement x as a function of time t. (2 marks)

Solution
We have $v = \frac{dx}{dt} = \frac{x}{8}$ $\therefore \int \frac{1}{x}dx = \int \frac{1}{8}dt$ using separation of variables   i.e. $\ln x = \frac{t}{8} + c_1$ Using $x(0) = 192$ $\therefore \ln(192) = c$ $\therefore \ln\left(\frac{x}{192}\right) = \frac{t}{8}$ $\therefore \frac{x}{192} = e^{\frac{t}{8}}$ *i.e.* $x(t) = 192\, e^{\frac{t}{8}}$
Specific behaviours
✓ formulates the differential equation $\frac{dx}{dt} = \frac{x}{8}$ correctly   ✓ determines the function $x(t)$ correctly

Question 11 (continued)

(c) Calculate the top speed reached and the distance travelled during the 11 second period of acceleration. (4 marks)

Solution
$x(t) = 192\,e^{\frac{t}{8}} \quad \therefore \quad v(t) = 24e^{\frac{t}{8}}$ $v(11) = 24e^{\frac{11}{8}} = 94.921...$ m/sec Distance travelled $= \Delta x \quad (\text{since } v > 0)$ $= \int_0^{11} v(t)\,dt = x(11) - x(0)$ $= 567.374...$ m Hence top speed is 95 m/sec and it travels 567 m during this acceleration.
Specific behaviours
✓ determines the velocity function $v(t)$ correctly ✓ calculates the top speed correctly ✓ forms the expression for the distance travelled correctly ✓ calculates the distance travelled correctly

Question 12 **(9 marks)**

The inner diameter of a cylinder in a motor car engine is critical to its performance. Let μ mm denote the population mean cylinder diameter produced by a manufacturing process. A random sample, R_1, of 100 cylinder diameters is taken and the standard deviation for this sample was found to be 1 mm.

Let $\overline{X} =$ the sample mean cylinder diameter for sample R_1.

(a) State the distribution for $\overline{X}$ and its parameters. (3 marks)

Solution
Since $n = 100 > 30$, then $\overline{X} \sim N\left(\mu,\ \sigma_{\overline{X}}^{\ 2}\right)$ i.e. normally distributed and centred with a mean μ and estimated standard deviation of the sample mean $\sigma\left(\overline{X}\right) = \dfrac{1}{\sqrt{100}} = 0.1$ mm.
Specific behaviours
✓ states that the sample mean will be normally distributed ✓ states the mean of the distribution is μ ✓ states the expected standard deviation is 0.1 mm

(b) What is the probability that $\overline{X}$ differs from μ by more than 0.2 mm. Give your answer correct to 0.001. (2 marks)

Solution
$P\left(\left\lvert\overline{X} - \mu\right\rvert > 0.2\right) = P\left(\lvert z\rvert > 2\right) = 2\left(0.023\right) = 0.046$
Specific behaviours
✓ forms the correct probability statement in terms of z ✓ calculates the probability correct to 0.001

From random sample R_1, a 95% confidence interval for μ is formed.

(c) Calculate the width of this confidence interval, correct to 0.001. (2 marks)

Solution
Width $w = 2\left(k\right)\left(\sigma\left(\overline{X}\right)\right) = 2\left(1.96\right)\left(0.1\right) = 0.392$
Specific behaviours
✓ forms the correct expression for the width ✓ calculates the width correctly

Question 12 (continued)

Lilian, the production manager, wishes to decrease the width of the confidence interval. She suggests:

"We can form sample R_2 by using the data from sample R_1 and then combining this data with itself to form a sample with 200 observations. Using $n = 200$ will decrease the width of the confidence interval."

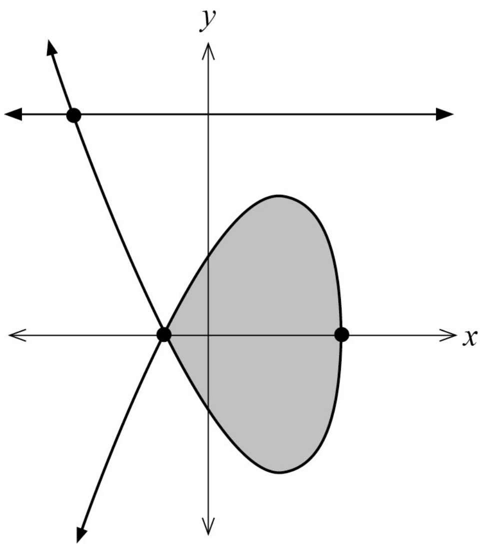
(d) State **two** major problems with using this idea. (2 marks)

<table>
<tr><th>Solution</th></tr>
<tr><td>1. The idea simply replicates (repeats) the data in sample R_1 . As such the sample R_2 is no longer random. Therefore the assumptions for using the normal distribution for the sample mean does not hold anymore.
2. Repeating the data values in sample R_1 will not reflect the true random variation in the data (manufacturing process). The confidence interval will therefore not be a true representation of the variation in the sample mean.
3. The sample mean and standard deviation will change if a new larger sample is taken. However, with Lillian's idea these will not change. This will affect both the width and location of the confidence interval.
4. Replicating the sample does NOT decrease the width of the confidence interval. Let the sample observations for R_1 be $x_1, x_2, \dots, x_{100}$.
$\overline{X_2} = \dfrac{\sum_{i=1}^{100} 2x_i}{200} = \overline{X_1}$ similarly $s_1^2 = s_2^2$ so $\overline{X_2} \sim N\left(\mu,\ 0.1^2\right)$.</td></tr>
<tr><th>Specific behaviours</th></tr>
<tr><td>✓ states that the R_2 data will no longer be a random sample
✓ states that the assumptions for using the normal distribution for the sample mean does not hold anymore

i.e. states any two of the four points outlined in the solution.</td></tr>
</table>

Question 13 **(8 marks)**

The equation $x^3 - x^2 - 5x = 3 - y^2$ implicitly defines the curve shown below. The line $y = \sqrt{24}$ intersects this curve as shown below.



It can be shown that the equation $x^3 - x^2 - 5x + 21 = 0$ will determine the intersection between the line $y = \sqrt{24}$ and the implicitly defined curve.

(a) Explain, with reference to the graph above, why we know that there is one real and two complex solutions (a conjugate pair) to this cubic equation. (2 marks)

Solution
There is only ONE point of intersection between the line $y = \sqrt{24}$ and the curve $(x = -3)$. Since the equation is a cubic with real coefficients, then the other two solutions must be complex (a conjugate pair).
Specific behaviours
✓ states there is just ONE point of intersection between $y = \sqrt{24}$ and the curve ✓ states this equation is cubic with real coefficients

(b) Determine the **two** exact complex solutions to the equation $x^3 - x^2 - 5x + 21 = 0$. (2 marks)

Solution
$x = -3$ is the real solution. Hence $(x+3)(x^2 - 4x + 7) = 0$ $\therefore\ x = -3$ or $(x-2)^2 + 3 = 0$ $\therefore\ x = 2 \pm \sqrt{3}\,i$ are the two complex solutions
Specific behaviours
✓ states that $x = -3$ is the real solution OR states that $(x+3)$ is a factor ✓ obtains the two complex solutions correctly

Question 13 (continued)

(c) Calculate the area of the shaded region, correct to 0.001 square units. (4 marks)

Solution
Intersection with the x axis: Solve $x^3 - x^2 - 5x = 3 - (0)^2$ Hence x intercepts are $x = -1,\ x = 3$ Equation: $y^2 = 3 - \left(x^3 - x^2 - 5x\right) \quad \therefore \ y = \pm\sqrt{3 - \left(x^3 - x^2 - 5x\right)}$ Area = $2\int_{-1}^{3} y\,dx = 2\int_{-1}^{3} \sqrt{3 - \left(x^3 - x^2 - 5x\right)}\,dx = 2(8.5333...) = 17.067$
Specific behaviours
✓ forms a definite integral with respect to x using the correct limits ✓ writes a factor of 2 in the integrand due to the symmetry about $y = 0$ ✓ writes the correct integrand for the y value in terms of x ✓ calculates the area correctly

Question 14 **(6 marks)**

The annual incomes (in thousands of dollars) of a random sample of n Australians is taken. The sample standard deviation is 10.98. A 99% confidence interval I_1 based on this sample is $90 \le \mu \le 94$.

(a) Calculate the value of the sample size n. (2 marks)

Solution
For 99% confidence using $k = 2.576$
$2 = (2.576)\left(\dfrac{10.98}{\sqrt{n}}\right)$ Solving gives $n = 200.0029....$ i.e. $n = 200$
OR $n = \left(\dfrac{k \times s}{w}\right)^2 = \left(\dfrac{2.576 \times 10.98}{2}\right)^2 = 200.0029...$
Note: Using $k = 2.58$ gives $n = 200.6245....$ i.e. $n = 201$
Specific behaviours
✓ uses the correct half-width of the interval to correctly form the equation for n
✓ determines the sample size as an integer

Another random sample of size n is taken and a 99% confidence interval I_2 is calculated.

(b) State **two** aspects in which the intervals I_1 and I_2 may be different. (2 marks)

Solution
1. The intervals may have different midpoints, as the sample means from the two sample may be different. 2. The intervals may have different widths, as the sample standard deviations may be different.
Specific behaviours
✓ states the interval midpoints may be different due to the different sample means
✓ states the interval widths may be different due to the different sample standard deviations

Question 14 (continued)

A third random sample of size 50 is taken and a 99% confidence interval I_3 is calculated. James suggests that since interval I_3 is the widest, it is more likely to contain the population mean Australian income μ.

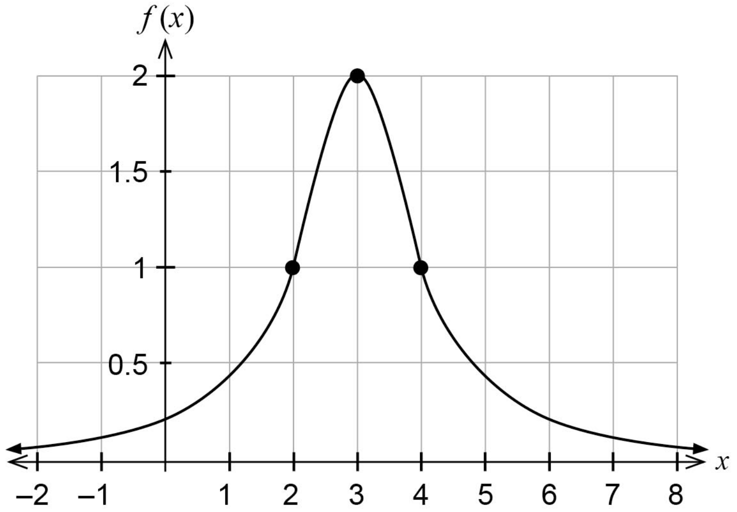
(c) Is James correct? Justify your answer. (2 marks)

Solution
No James is NOT correct. The interval is wider because it is based on data with more variation in it. Consequently, the sample mean has more variation. That is, the location of the interval is more variable. We cannot be sure that any confidence interval contains the true mean.
Specific behaviours
✓ states that James is NOT correct ✓ provides correct justification (we cannot be certain that any confidence interval contains the mean)

Question 15 **(4 marks)**

The graph of a rational function f is shown below. Function f has the form $f(x) = \dfrac{k}{q(x)}$ with the following properties:

- f has no x intercepts or vertical asymptotes
- $f(x) \to 0$ for $|x| \to \infty$
- f is symmetric about $x = 3$
- function q is quadratic
- k is a constant



Determine the defining rule for f.

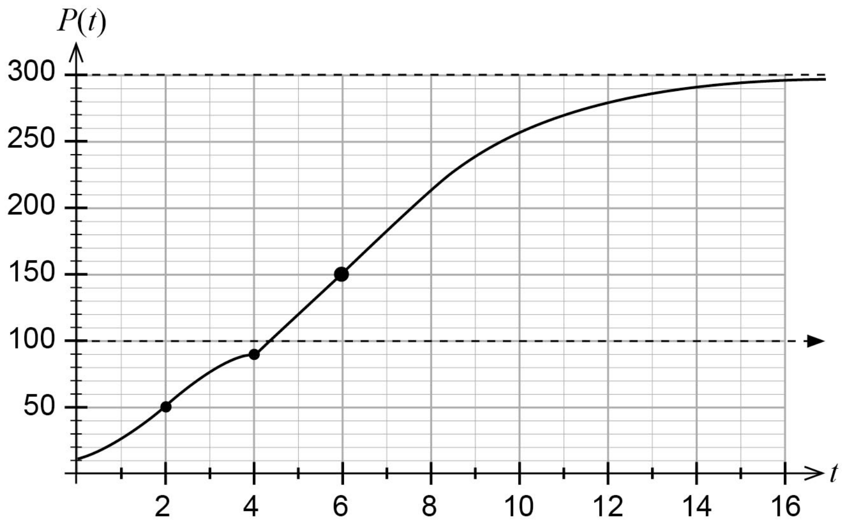
Solution
Since q is quadratic with symmetry about $x = 3$ $q(x) = a(x-3)^2 + b$
As there are no vertical asymptotes then $q(x) \neq 0$ $\therefore$ $a, b > 0$ using $k > 0$.
The maximum of f will occur when q is a minimum.
Using $f(3) = 2$ $\quad 2 = \dfrac{k}{a(3-3)^2 + b}$ $\quad$ *i.e.* $k = 2b$ ... (1)
Using $f(2) = 1$ $\quad 1 = \dfrac{k}{a(2-3)^2 + b}$ $\quad$ *i.e.* $a + b = k$ ... (2)
Solving $(1),(2)$ simultaneously we obtain $a = b,\ k = 2b$
i.e. $f(x) = \dfrac{2b}{b(x-3)^2 + b} = \dfrac{2}{(x-3)^2 + 1} = \dfrac{2}{x^2 - 6x + 10}$
Specific behaviours
✓ forms a quadratic rule for $q(x)$ that has symmetry about $x = 3$
✓ forms a quadratic rule for $q(x)$ that does NOT have any x intercepts $(a, b > 0)$
✓ forms correct relationships between k and other constants
✓ determines the correct defining rule for $f(x)$

Question 15 (continued)

Alternative Solution
Since q is quadratic then let $q(x) = ax^2 + bx + c$ Using $f(3)=2 \quad 2 = \dfrac{k}{9a+3b+c} \quad \ldots (1)$ $f(2)=1 \quad 1 = \dfrac{k}{4a+2b+c} \quad \ldots (2)$ $f(4)=1 \quad 1 = \dfrac{k}{16a+4b+c} \quad \ldots (3)$ Solving $(1),(2),(3)$, simultaneously we obtain $a=a, \quad b=-6a, \quad c=10a, \quad k=2a$ i.e. $f(x) = \dfrac{2a}{ax^2-6ax+10a} = \dfrac{2a}{a(x^2-6x+10)} = \dfrac{2}{x^2-6x+10}$
Specific behaviours
✓ uses the rule $q(x) = ax^2 + bx + c$ and $f(3)=2, f(2)=f(4)=1$ to form three equations relating k,a,b,c ✓ solves simultaneously to obtain relationships between variables ✓ factors out the common factor between variables ✓ determines the correct defining rule for $f(x)$

Question 16 **(4 marks)**

An ant colony population P at time t days grows at a rate given by the equation $\frac{dP}{dt} = 0.01P(100-P)$, where $0 \le t \le 4$. The graph of this population is shown below.



(a) For $0 \le t \le 4$, using the growth rate equation explain the variation of the population. (2 marks)

Solution
Initially when $P < 50$ the growth rate increases i.e. P curve is concave up. When $P > 50$ the growth rate decreases i.e. P curve is concave down. Hence there is a point of inflection when $P = 50$ and then the curve will approach the horizontal asymptote $P = L$.
Specific behaviours
✓ states that the slope (growth rate) increases when $P < 50$ or $t < 2$ ✓ states that the slope (growth rate) decreases when $P > 50$ or $t > 2$

At the end of the fourth day, the environment for the ant colony improves dramatically so that its limiting population is increased to 300.

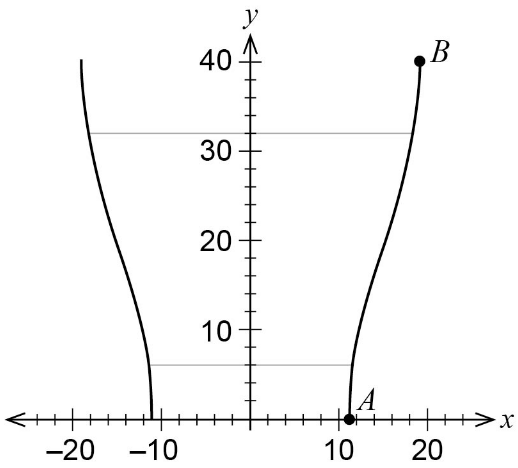
(b) Sketch, on the axes above, the expected variation of the population for $t > 4$ days, using the increased limiting population value. (2 marks)

Solution
Shown above
Specific behaviours
✓ indicates a point of inflection in the graph at $P = 150$ (value of t not important) ✓ indicates the curve flattens out as $P \to 300$

Question 17 **(10 marks)**

A vase has water with a depth of 6 cm and needs to be filled to a depth of 32 cm. The cross-section of the vase is modelled by the curve AB where $x = 15 - 4\cos\left(\frac{\pi y}{40}\right)$, $0 \le y \le 40$, and this curve is revolved about the y axis.
All dimensions are in centimetres.

Give all answers in this question to the nearest appropriate unit of measurement.



(a) Calculate the volume of water that needs to be added to increase the depth of water from 6 cm to 32 cm. (3 marks)

Solution
Volume water required $= \int_6^{32} \pi\left(15 - 4\cos\left(\frac{\pi y}{40}\right)\right)^2 dy = \pi(5763.9368)$ $= 18\ 107.9417...$ cm^3 Hence the volume of water required is 18 108 cm^3 (nearest cm^3). Accept 18 litres (nearest litre) provided the calculated value 18 107.94 … is shown.
Specific behaviours
✓ forms a definite integral with the correct limits and using correct notation   ✓ uses the correct integrand in terms of y including the factor of π ✓ calculates the correct volume (to nearest unit) using the correct units

Josie, an interior designer, uses a hose to add water to the vase. This hose has a water-saving device that regulates the rate at which water flows into the vase. This rate is given by:

$$\frac{dV}{dt} = 300e^{-\frac{V}{12\,000}}$$

where $V(t) =$ the volume of water (cm^3) poured into the vase after t seconds of flow.

(b) If Josie has already poured 6000 cm^3, use the increments formula to calculate an approximation for the volume of water she will pour in the next 0.5 seconds. (2 marks)

Solution
$\Delta V \simeq \left(\frac{dV}{dt}\right) \times \Delta t = \left(300e^{-\frac{6000}{12000}}\right) \times 0.5$ $= (181.959..) \times 0.5 = 90.979...$ cm^3 Hence she will pour approx. 91 cm^3 (nearest cm^3) in the next 0.5 seconds.
Specific behaviours
✓ substitutes $V = 6000$ and $\Delta t = 0.5$ correctly into the increments formula   ✓ calculates the correct volume

To prevent an overflow of water, the device can be calibrated to switch off the flow after a set length of time using an in-built timer.

(c) Calculate the rate of flow into the vase at the instant when the depth becomes 32 cm. (2 marks)

Solution
Require $\frac{dV}{dt}$ when $V = 18\,107.9417$ cm³   i.e. $\frac{dV}{dt} = 300e^{-\frac{18\,107.9417}{12\,000}} = 66.3396$ cm³/sec   The rate of flow when the depth is 32 cm is approx. 66 cm³/sec.
Specific behaviours
✓ substitutes the total volume from part (a) into the differential equation   ✓ calculates the rate correctly stating the correct units

Alternative Solution
Solve for when $V(t) = 18\,107.9417...$ This gives $t = 140.8873...$ sec $V'(t) = 12000 \times \frac{1}{\left(\frac{t}{40}+1\right)} \times \frac{1}{40} = \frac{300}{\left(\frac{t}{40}+1\right)}$ Evaluating $V'(140.8873...) = 66.3396...$ cm³/sec   The rate of flow when the depth is 32 cm is approx. 66 cm³/sec.
Specific behaviours
✓ solves correctly the value of t that gives the volume obtained in part (a)   ✓ calculates the rate correctly stating the correct units

(d) Using separation of variables, obtain the defining rule for $V(t)$. (3 marks)

Solution
$\int e^{\frac{V}{12\,000}}\,dV = \int 300\,dt$ $12\,000\,e^{\frac{V}{12\,000}} = 300t + c$ Using $V(0)=0$ $12\,000e^{0} = 300(0) + c$ $\therefore\ c = 12\,000$ $12\,000\,e^{\frac{V}{12\,000}} = 300t + 12\,000$ $\therefore\ e^{\frac{V}{12\,000}} = \frac{t}{40} + 1$ $\therefore\ \frac{V}{12000} = \ln\left(\frac{t}{40} + 1\right)$ *i.e.* $V(t) = 12\,000\ln\left(\frac{t}{40} + 1\right)$
Specific behaviours
✓ separates the variables as an integration statement correctly   ✓ anti-differentiates correctly using a constant   ✓ determines V as the subject correctly from the anti-derivative statement

Question 18 **(7 marks)**

(a) Show that for all positive integers n and complex numbers z where $0 \leq \theta \leq \frac{\pi}{2}$,

$\left(z^n - cis(\theta)\right)\left(z^n + cis(-\theta)\right) = z^{2n} - (2i\sin\theta)z^n - 1$. (3 marks)

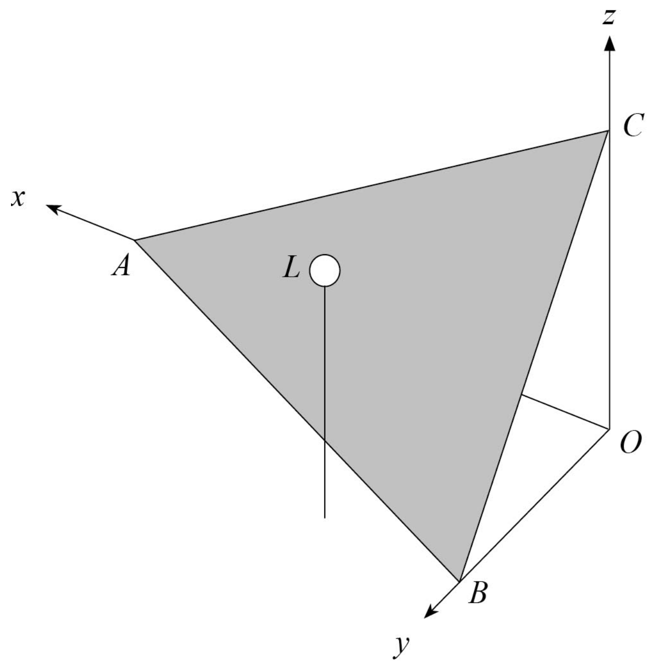
Solution
$\left(z^n - cis(\theta)\right)\left(z^n + cis(-\theta)\right)$ $= z^{2n} - cis(\theta)z^n + cis(-\theta)z^n - cis(\theta)cis(-\theta)$ $= z^{2n} - \left(cis(\theta) - cis(-\theta)\right)z^n - cis(0)$ $= z^{2n} - (2i\sin\theta)z^n - 1$
Specific behaviours
✓ expands the binomial products correctly to obtain 4 terms ✓ uses the property $cis(\theta)cis(-\theta) = cis(0) = 1$ ✓ uses the property $cis(\theta) - cis(-\theta) = 2i\sin\theta$

(b) Hence, using the result from part (a), obtain all the solutions to the equation $z^6 - (i)z^3 - 1 = 0$ in exact polar form. (4 marks)

Solution
To solve $z^6 - (i)z^3 - 1 = 0$ i.e. $\left(z^3\right)^2 - \left(2\left(\frac{1}{2}\right)i\right)z^3 - 1 = 0$ $\therefore\ n = 3$ and $\sin\theta = \frac{1}{2}$ i.e. $\theta = \frac{\pi}{6}$ using $0 \le \theta \le \frac{\pi}{2}$ from part (a). Hence solve $\left(z^3 - cis\left(\frac{\pi}{6}\right)\right)\left(z^3 + cis\left(-\frac{\pi}{6}\right)\right) = 0$ $\therefore\ z^3 = cis\left(\frac{\pi}{6}\right) \quad \therefore\ z = cis\left(\frac{\pi}{18}\right), cis\left(\frac{13\pi}{18}\right), cis\left(\frac{25\pi}{18}\right)$ or $cis\left(-\frac{11\pi}{18}\right)$ OR $z^3 = -cis\left(-\frac{\pi}{6}\right) = cis(\pi)cis\left(-\frac{\pi}{6}\right) = cis\left(\frac{5\pi}{6}\right)$ $\therefore\ z = cis\left(\frac{5\pi}{18}\right), cis\left(\frac{17\pi}{18}\right), cis\left(\frac{29\pi}{18}\right)$ or $cis\left(-\frac{7\pi}{18}\right)$
Specific behaviours
✓ determines the value of n and θ correctly ✓ deduces that $z^3 = cis\left(\frac{\pi}{6}\right)$ and $z^3 = cis\left(\frac{5\pi}{6}\right)$ ✓ gives ALL solutions for $z^3 = cis\left(\frac{\pi}{6}\right)$ correctly ✓ gives ALL solutions for $z^3 = cis\left(\frac{5\pi}{6}\right)$ correctly

Question 19 **(9 marks)**

A downward-sloping ramp is positioned according to the coordinate system shown. $A(6,0,0)$, $B(0,2,0)$ and $C(0,0,3)$ are points on the ramp. A lamp L is positioned on top of a post at $\left(2,\ 2,\ \frac{5}{2}\right)$. All dimensions are measured in metres.



(a) Determine the Cartesian equation for the ramp. (2 marks)

Solution
Cartesian equation of a plane is $ax+by+cz=d$ Using the ordered pairs we obtain equations: $6a=d$, $2b=d$ and $3c=d$. By choosing $d=6$ we obtain $a=1,\ b=3,\ c=2$ $\therefore$ Equation for the ramp is given by $x+3y+2z=6$ (where $x,y,z\geq 0$)
Specific behaviours
✓ substitutes correctly into the equation $ax+by+cz=d$ to form 3 equations ✓ selects a suitable value for d and solves for the coefficients $a,\ b,\ c$

Alternative Solution
Normal vector can be given by the cross product of any 2 vectors that lie in the plane e.g. $\underset{\sim}{n} = \overrightarrow{BA} \times \overrightarrow{BC} = \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ -18 \\ -12 \end{pmatrix} = -6\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ Vector equation for plane: $\underset{\sim}{r} \bullet \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = 6$ i.e. $x+3y+2z=6$
Specific behaviours
✓ determines the normal vector for the plane correctly ✓ forms the Cartesian equation for the plane correctly

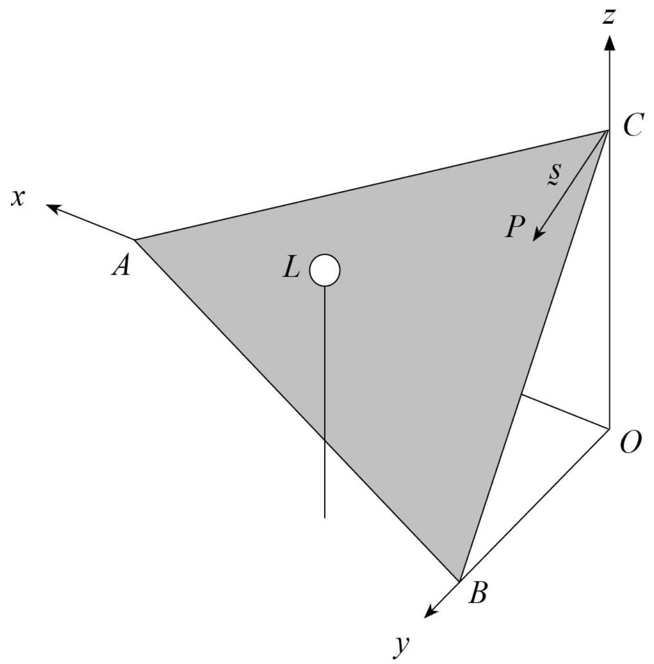
At night, the lamp L emits a bright light and illuminates the ramp. The position that is closest to the lamp will be the most brightly illuminated.

(b) Determine the coordinates for the point on the ramp that is the most brightly illuminated. (4 marks)

Solution
The point that is illuminates the most is the point on the plane that is CLOSEST to the point L. This will be on the PERPENDICULAR to the plane from L. Perpendicular line to plane: $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 2 \\ 2.5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2+\lambda \\ 2+3\lambda \\ 2.5+2\lambda \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ Intersection with plane: $(2+\lambda) + 3(2+3\lambda) + 2(2.5+2\lambda) = 6$ i.e. $13 + 14\lambda = 6$ $\therefore \quad \lambda = -0.5$ Hence point that is illuminated the most will be $(1.5,\ 0.5,\ 1.5)$.
Specific behaviours
✓ determines the equation for the line normal to the plane containing L ✓ substitutes correctly into the equation of the plane to solve simultaneously ✓ solves for the value of the parameter λ correctly ✓ states the coordinates for the most illuminated point correctly

Question 19 (continued)

If a ball is released from point C and is allowed to roll down the ramp, gravity will cause it to follow the path of steepest descent. Suppose the ball is allowed to roll exactly 1 metre from point C to point P, where $\underset{\sim}{s} = \overrightarrow{CP}$ is the direction of the steepest descent down the ramp.



(c) Determine vector $\underset{\sim}{s}$, giving components correct to 0.001. (3 marks)

Solution
Steepest descent will be determined by the path that goes from C to a point G where G is a point of $\overline{AB}$ where $\overrightarrow{CG} \perp \overline{AB}$. Note that $\underset{\sim}{s} = k\,\overrightarrow{CG}$. Let $\overrightarrow{CG}$ have direction vector $\underset{\sim}{d}$ where $\underset{\sim}{d} \perp \overline{AB}$ and $\underset{\sim}{d} \perp \underset{\sim}{n}$ i.e. $\underset{\sim}{d} = \overrightarrow{AB} \times \underset{\sim}{n}$ $\therefore\ \underset{\sim}{d} = \begin{pmatrix}-6\\2\\0\end{pmatrix} \times \begin{pmatrix}1\\3\\2\end{pmatrix} = \begin{pmatrix}4\\12\\-20\end{pmatrix} = 4\begin{pmatrix}1\\3\\-5\end{pmatrix}$ We require $\lvert\underset{\sim}{s}\rvert = 1$ hence $k\left\lVert\begin{pmatrix}1\\3\\-5\end{pmatrix}\right\rVert = 1$ i.e. $k\left(\sqrt{35}\right) = 1$ $\therefore\ k = \frac{1}{\sqrt{35}}$ Hence $\underset{\sim}{s} = \left(\frac{1}{\sqrt{35}}, \frac{3}{\sqrt{35}}, -\frac{5}{\sqrt{35}}\right)$ i.e. $\underset{\sim}{s} = (0.169,\ 0.507,\ -0.845)$ (3 d.p.) Note: Greatest descent is 0.845 metres.
Specific behaviours
✓ states that the direction of $\underset{\sim}{s}$ is in the direction of the perpendicular to $\overline{AB}$ ✓ determines the correct direction for $\underset{\sim}{d}$ (not the unit vector) ✓ determines the components for $\underset{\sim}{s}$

Alternative Solution
Steepest descent will be determined by the path that goes from C to a point G where G is a point of $\overline{AB}$ where $\overrightarrow{CG} \perp \overline{AB}$. Note that $\underset{\sim}{s} = k\,\overrightarrow{CG}$. Equation for any position on $\overrightarrow{AB}$: $\underset{\sim}{r} = \begin{pmatrix}6\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}-6\\2\\0\end{pmatrix} = \begin{pmatrix}6-6\lambda\\2\lambda\\0\end{pmatrix}$ $\overrightarrow{CG} = \begin{pmatrix}6-6\lambda\\2\lambda\\0\end{pmatrix} - \begin{pmatrix}0\\0\\3\end{pmatrix} = \begin{pmatrix}6-6\lambda\\2\lambda\\-3\end{pmatrix}$ To determine the position for G we require $\overrightarrow{CG} \perp \overrightarrow{AB}$ i.e. $\overrightarrow{CG} \bullet \overrightarrow{AB} = 0$ $\begin{pmatrix}6-6\lambda\\2\lambda\\-3\end{pmatrix} \bullet \begin{pmatrix}-6\\2\\0\end{pmatrix} = 0$ $\therefore\ -6(6-6\lambda) + 2(2\lambda) + (-3)(0) = 0$ Solving gives $\lambda = 0.9$ $\therefore$ G has position $\begin{pmatrix}6-6(0.9)\\2(0.9)\\0\end{pmatrix} = \begin{pmatrix}0.6\\1.8\\0\end{pmatrix}$ Hence $k\,\overrightarrow{CG} = \begin{pmatrix}0.6\\1.8\\-3\end{pmatrix} = \underset{\sim}{s}$ We require $\lvert\underset{\sim}{s}\rvert = 1$ hence $k\left\lvert\begin{pmatrix}0.6\\1.8\\-3\end{pmatrix}\right\rvert = 1$ i.e. $k\left(\sqrt{12.6}\right) = 1$ $\therefore$ $k = \dfrac{1}{\sqrt{12.6}}$ Hence $\underset{\sim}{s} = \dfrac{1}{\sqrt{12.6}}(0.6,\ 1.8,\ -3) = (0.169,\ 0.507,\ -0.845)$ (3 d.p.) Note: Greatest descent is 0.845 metres.
Specific behaviours
✓ states that the direction of $\underset{\sim}{s}$ is in the direction of the perpendicular to $\overline{AB}$ ✓ determines vector $\overrightarrow{CG}$ (not the unit vector) ✓ determines the components for $\underset{\sim}{s}$

Question 19 (continued)

Alternative Solution

Vector $\underset{\sim}{s} = \overrightarrow{CP} = (a,b,c)-(0,0,3) = (a,b,\ c-3)$

Since P is a point in the plane then $a+3b+2c=6$

i.e. $c = -\frac{a}{2}-\frac{3b}{2}+3 \quad \therefore \ \underset{\sim}{s} = \left(a,b,\ -\left(\frac{a}{2}+\frac{3b}{2}\right)\right)$

We also have that $|\underset{\sim}{s}|=1$

$\therefore \ a^2+b^2+\left(\frac{a}{2}+\frac{3b}{2}\right)^2 = 1 \quad \ldots (1)$

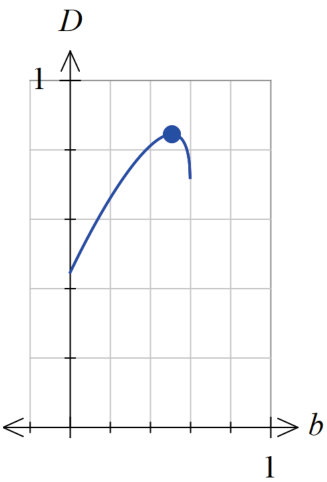
$\therefore \ 5a^2+6ab+(13b^2-4)=0$

Obtaining a in terms of b: $a = \dfrac{-3b + 2\sqrt{5-14b^2}}{5}$

The DESCENT $D = 3-c$ (the change in the z coordinates from $C \to P$)

i.e. $D = \frac{a}{2} + \frac{3b}{2} = \frac{1}{2}\left(\frac{-3b + 2\sqrt{5-14b^2}}{5}\right) + \frac{3b}{2} \quad \ldots (2)$

Plotting the function D versus b:



Maximum T.P. at $(0.50709\ldots,\ 0.84515\ldots)$

i.e. there is a maximum when $b=0.50709\ldots$ that gives a maximum DESCENT $d=0.845$ metres.

$a=0.16903\ldots$
$c=-0.84515\ldots$

Hence $\underset{\sim}{s} = \begin{pmatrix} 0.169 \\ 0.507 \\ -0.845 \end{pmatrix}$ (3 d.p.)

(Notice that $b=3a$ for the steepest descent)

Specific behaviours

- ✓ forms equation (1) in terms of a,b from $|\underset{\sim}{s}|=1$
- ✓ forms the function (2) for the descent d in terms of one variable
- ✓ determines the vector components a,b,c

Published by the School Curriculum and Standards Authority of Western Australia
303 Sevenoaks Street
CANNINGTON WA 6107