Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS METHODS

Calculator-assumed

ATAR course examination 2023

Marking key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

Section Two: Calculator-assumed **65% (96 Marks)**

Question 6 **(11 marks)**

(a) Determine the initial population of the bee colony. (1 mark)

Solution
$B = 4000$ bees
Specific behaviours
✓ correctly determines initial population

(b) Determine the increase in the population of the bee colony in the first six months. (2 marks)

Solution
$B(0.5) = 4e^{1.4(0.5)}$ $\approx 8.055...$ Population increase ≈ 8055 – 4000 = 4055
Specific behaviours
✓ correctly calculates $B(0.5)$ ✓ correctly calculates increase in bee population

(c) Determine the rate of population growth two years after the establishment of the colony. (2 marks)

Solution
$\frac{dB}{dt} = 5.6e^{1.4t}$ $\left.\frac{dB}{dt}\right\|_{t=2} = 5.6e^{1.4(2)}$ $= 92.09...$ Hence the rate of population growth two years after the establishment of the colony is approximately 92 000 bees per year.
Specific behaviours
✓ correctly differentiates population equation   ✓ correctly determines rate of population growth

(d) After how many years will the rate of population growth be 65 000 bees/year? (2 marks)

Solution
$65 = 5.6e^{1.4t}$ $t = 1.751...$ Hence the rate of population growth will be 65 000 bees/year after 1.75 years
Specific behaviours
✓ correctly substitutes $B = 65$ into growth rate equation   ✓ correctly determines number of years

(e) Determine A and r if one year after the start of the decline the bee population is 100 000. (4 marks)

Solution
$B(3) = 4e^{1.4(3)}$ $= 266.745...$ $\therefore A = 267$ $100 = 267e^{r(1)}$ $\therefore r = -0.982...$ $\therefore b(t) = 267e^{-0.98t}$
Specific behaviours
✓ correctly determines population after 3 years of growth   ✓ identifies value of A ✓ correctly substitutes value of A and $b = 100$ into equation to solve for r ✓ correctly solves for r

Question 7 **(9 marks)**

(a) On the basis of the sample, determine a point estimate for p. (1 mark)

Solution
$\hat{p} = \frac{76}{200} = 0.38$
Specific behaviours
✓ correctly calculates the sample proportion

(b) On the basis of the sample, determine a 95% confidence interval for p. (2 marks)

Solution
$95\%\ \text{CI} = \left(0.38 - 1.96\sqrt{\frac{0.38(1-0.38)}{200}},\ 0.38 + 1.96\sqrt{\frac{0.38(1-0.38)}{200}}\right) = (0.3127,\ 0.4473)$
Specific behaviours
✓ uses correct critical value from the normal distribution ✓ calculates confidence interval correctly

(c) What is the minimum number of birds that would need to be sampled to ensure that the margin of error of the 95% confidence interval for p is at most 0.02? (2 marks)

Solution
The margin of error will be maximised when $\hat{p} = 0.5$. Hence $0.02 = 1.96\sqrt{\frac{0.5(1-0.5)}{n}}$ $\Rightarrow n = 2401$
Specific behaviours
✓ uses $\hat{p} = 0.5$ to consider the maximum margin of error ✓ calculates the correct sample size

(d) Identify and explain **two** sources of bias in the birdwatcher's sampling method. (4 marks)

Solution
Answers could include: • Single location: Different bird species are likely to be clustered around areas of the national park that provide them with a suitable habitat (i.e. food, vegetation, nesting sites etc). By selecting a single location, bird species who prefer the local habitat are more likely to be observed/included in the sample. • Single time: Different bird species may be more/less active at particular times of the day and/or at particular times of the year. Birds that are less active at the time/day the birdwatcher was observing are less likely to be included in the sample. • Variety of behaviours: Different bird species fly at different altitudes (or not at all), have varying sizes, have varying degrees of camouflage etc. Hence, birds that are more noticeable/stand out more in the birdwatcher's field of view are more likely to be included in the sample.
Specific behaviours
✓ identifies a source of bias ✓ explains how the source introduces bias ✓ identifies a second source of bias ✓ explains how the second source introduces bias

Question 8 **(12 marks)**

(a) Determine the first two times, $t > 0$, at which the mass changes direction. State your answers exactly. (2 marks)

Solution
The mass changes direction when $v(t) = 0$. From the graph it is clear that $v(t) = 0$ when $t = \pi$ seconds, and $t = 2\pi$ seconds.
Specific behaviours
✓ states that a change in direction occurs when $v(t) = 0$ ✓ correctly determines the first two times the mass changes direction

(b) What does the signed area of the shaded region in the figure represent? (2 marks)

Solution
The shaded region represents the change in the displacement of the mass from $t = 0$ to $t = \frac{5\pi}{3}$ seconds.
Specific behaviours
✓ states that the shaded region represents a change in displacement ✓ specifies the correct start and finish time

(c) Write an integral expression for the distance travelled from $t = \frac{\pi}{3}$ to $t = \frac{4\pi}{3}$. (3 marks)

Solution
distance = $\int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} \left| 2t\cos\left(t + \frac{\pi}{2}\right) \right| dt$
Specific behaviours
✓ identifies distance as the integral of speed (absolute value of velocity) ✓ writes the correct integral (including bounds) ✓ dt included

Alternative solution one
distance = $-\int_{\frac{\pi}{3}}^{\pi} 2t\cos\left(t + \frac{\pi}{2}\right) dt + \int_{\pi}^{\frac{4\pi}{3}} 2t\cos\left(t + \frac{\pi}{2}\right) dt$
Specific behaviours
✓ split into two integrals with correct bounds and correct integrand ✓ correct sign (+/-) in front of integrals ✓ dt included in both integrals

Alternative solution two
distance $= \left\| \int_{\frac{\pi}{3}}^{\pi} 2t\cos\left(t+\frac{\pi}{2}\right)dt \right\| + \left\| \int_{\pi}^{\frac{4\pi}{3}} 2t\cos\left(t+\frac{\pi}{2}\right)dt \right\|$
Specific behaviours
✓ split into two integrals with correct bounds and correct integrand ✓ absolute value signs included (either inside the integral or outside) ✓ dt included in both integrals

(d) Determine the first time after $t = \pi$ that the acceleration of the object will be 0 m/s^2. (2 marks)

Solution
The acceleration is given by $a(t) = v'(t) = 2\cos\left(t+\frac{\pi}{2}\right) - 2t\sin\left(t+\frac{\pi}{2}\right)$ or $a(t) = v'(t) = -2\sin(t) - 2t\cos(t)$ Solving $v'(t) = 0$ gives $t \approx 4.91$ seconds
Specific behaviours
✓ states correct expression for acceleration ✓ solves to obtain correct time

(e) The displacement of the mass is given by

$$x(t) = A\sin\left(t+\frac{\pi}{2}\right) + B\cos\left(t+\frac{\pi}{2}\right) + 2t\sin\left(t+\frac{\pi}{2}\right)$$

where A and B are constants. Determine the value of A and B. (3 marks)

Solution
$v(t) = x'(t) = A\cos\left(t+\frac{\pi}{2}\right) - B\sin\left(t+\frac{\pi}{2}\right) + 2\sin\left(t+\frac{\pi}{2}\right) + 2t\cos\left(t+\frac{\pi}{2}\right)$ Given that $v(t) = 2t\cos\left(t+\frac{\pi}{2}\right)$ it follows that $A = 0$ and $-B + 2 = 0$ $\Rightarrow B = 2$
Specific behaviours
✓ correctly differentiates $x(t)$ ✓ compares $x'(t)$ and $v(t)$ to determine that $A = 0$ ✓ compares $x'(t)$ and $v(t)$ to determine that $B = 2$

Question 9 **(10 marks)**

(a) Determine the cross-sectional area of the building. (2 marks)

Solution
$\text{Area} = \int_0^{10\pi} \left(6\sin\left(\frac{w}{10}\right) + 3\sin\left(\frac{w}{5}\right)\right) dx$ $= 120 \text{ m}^2$
Specific behaviours
✓ states a correct integral expression for the cross-sectional area   ✓ correctly determines the cross-sectional area including units

(b) With reference to the figure

(i) determine the values of w_1 and w_2. (2 marks)

Solution
Solving $h(w) = g(w)$: $\Rightarrow 6\sin\left(\frac{w}{10}\right) + 3\sin\left(\frac{w}{5}\right) = 7\cos\left(\frac{w}{20}\right)$ $\Rightarrow w = 6.6511,\ 18.4122$ Hence $w_1 = 6.6511$ and $w_2 = 18.4122$.
Specific behaviours
✓ states correct equation to solve   ✓ determines the correct values of w_1 and w_2

(ii) determine the area of the window. (2 marks)

Solution
$\text{Area} = \int_{6.6511}^{18.4122} \left(6\sin\left(\frac{w}{10}\right) + 3\sin\left(\frac{w}{5}\right) - 7\cos\left(\frac{w}{20}\right)\right) dx$ $= 13.94 \text{ m}^2$
Specific behaviours
✓ states a correct integral expression for the cross-sectional area   ✓ correctly determines the cross-sectional area

(c) Use calculus techniques to determine the maximum height of the building. (4 marks)

Solution
The derivative of $h(w)$ is given by $$h'(w)=\frac{3}{5}\cos\left(\frac{w}{10}\right)+\frac{3}{5}\cos\left(\frac{w}{5}\right)$$ Setting $h'(w)=0$ yields $$w=\frac{10\pi}{3}\ \ (\approx 10.47)$$ The second derivative of $h(w)$ is $$h''(w)=-\frac{3}{50}\sin\left(\frac{w}{10}\right)-\frac{3}{25}\sin\left(\frac{w}{5}\right)$$ Since $h''\left(\frac{10\pi}{3}\right)=-\frac{9\sqrt{3}}{100}\approx -0.156<0$ it follows that $w=\frac{10\pi}{3}$ is a local maximum. The maximum height of the building is $$h\left(\frac{10\pi}{3}\right)=6\sin\left(\frac{\pi}{3}\right)+3\sin\left(\frac{2\pi}{3}\right)$$ $$=\frac{9\sqrt{3}}{2}\text{ m }(\approx 7.79\text{ m})$$
Specific behaviours
✓ states correct derivative for $h(w)$ ✓ sets $h'(w)=0$ and obtains correct critical value ✓ calculates $h''\left(\frac{10\pi}{3}\right)$ and concludes local maximum ✓ calculates correct maximum height

Question 10 **(7 marks)**

(a) On the basis of the diagram above, is it appropriate to use the normal distribution to approximate the distribution of $\hat{p}$? Justify your answer. (2 marks)

Solution
No. The distribution of $\hat{p}$ is not symmetric and so the normal approximation is not appropriate.
Specific behaviours
✓ states that the normal approximation is not appropriate ✓ provides appropriate justification

(b) Determine the probability that more than 30 people in the sample have arch-shaped fingerprints. (3 marks)

Solution
Let the random variable X denote the number of people in the sample with arch-shaped fingerprints. Then $X \sim \text{Bin}(500,\ 0.05)$ Hence $P(X > 30) = P(X \geq 31) = 0.1309$
Specific behaviours
✓ identifies binomial distribution with correct values for the parameters n and p ✓ correctly states $P(X > 30)$ or $P(X \geq 31)$ ✓ calculates correct probability

(c) Use the approximate normality of the distribution to determine the probability that the sample proportion of people with arch-shaped fingerprints is greater than 0.06. (2 marks)

Solution
Let $\hat{p}$ denote the sample proportion of people with arch-shaped fingerprints. Then $\hat{p} \sim N\left(0.05, \frac{0.05(1-0.05)}{500}\right)$ $\hat{p} \sim N(0.05, 0.000095)$ Hence $P(\hat{p} > 0.06) \approx 0.1525$
Specific behaviours
✓ states correct values for the distribution parameters mean and variance/standard deviation ✓ determines correct probability

Question 11 **(13 marks)**

(a) Determine the value of k. (2 marks)

Solution
The area under the curve must be equal to 1. Hence $4\times0.1+0.5\times2\times(k-0.1)=1$ $\Rightarrow k+0.3=1$ $\Rightarrow k=0.7$
Specific behaviours
✓ states that the area under the curve must equal 1 ✓ obtains correct value of k

(b) An incomplete expression for the probability density function of B is given below. Fill in the missing parts of the expression. (2 marks)

Solution
The probability density function for B is given by $f(b)=\begin{cases}0.1, & \mathbf{1\le b<3}\\ \mathbf{0.3b-0.8,} & 3\le b\le 5\\ 0, & \text{otherwise}\end{cases}$
Specific behaviours
✓ correctly completes the interval ✓ correctly completes the linear function

(c) Determine the expected time that Mrs Euler's vehicle will be ready for collection at BIMDAS Mechanics. (3 marks)

Solution
$E(B)=\int_1^3 0.1b\,db+\int_3^5 b(0.3b-0.8)\,db$ $=3.8$ Therefore, the expected pickup time is 3:48 pm.
Specific behaviours
✓ states a correct integral expression for the expected value of B ✓ determines the correct expected value of B ✓ states the expected value as a time

Question 11 (continued)

(d) Determine the probability that Mr Euler's vehicle will be ready to collect

(i) by 3 pm. (1 mark)

Solution
$$P(A \le 3) = \frac{10(3) - 3^2 - 9}{16} = 0.75$$
Specific behaviours
✓ calculates correct probability

(ii) between 3 pm and 4 pm. (2 marks)

Solution
$$P(3 \le A \le 4) = P(A \le 4) - P(A \le 3) = \frac{10(4) - 4^2 - 9}{16} - \frac{10(3) - 3^2 - 9}{16} = \frac{15}{16} - \frac{12}{16} = \frac{3}{16} = 0.1875$$
Specific behaviours
✓ expresses the probability as the difference $P(A \le 4) - P(A \le 3)$ ✓ calculates correct probability

(e) Determine the expected time at which Mr Euler's vehicle will be ready for collection at Addition Autos. (3 marks)

Solution
The probability density function is given by $$p(a) = \frac{d}{da}\left(\frac{10a - a^2 - 9}{16}\right) = \frac{5}{8} - \frac{a}{8}$$ for $1 \le a \le 5$ (0 otherwise). Hence the expected value is given by $$E(A) = \int_1^5 a\left(\frac{5}{8} - \frac{a}{8}\right)da = \frac{7}{3}\left(= 2\frac{1}{3}\right)$$ Therefore, the expected pickup time is 2:20 pm.
Specific behaviours
✓ determines correct expression for the probability density function for $1 \le a \le 5$ ✓ determines the correct expected value for A ✓ states the expected value as a time

Question 12 **(14 marks)**

(a) Determine the mean and standard deviation of the mass of noodles dispensed by the machine. (3 marks)

Solution
Given that $$P(Z \le -1.9991) = 0.0228$$ the Z-score corresponding to $X = 150$ is $Z = -1.9991$. Similarly, since $$P(Z \le 0.9998) = 0.1587$$ the Z-score corresponding to $X = 165$ is $Z = 0.9998$, it follows that the mean, μ, and standard deviation, σ, satisfy the equations $$-1.9991 = \frac{150-\mu}{\sigma}$$ and $$0.9998 = \frac{165-\mu}{\sigma}$$ which can be rearranged to give $$\mu - 1.9991\sigma = 150$$ $$\mu + 0.9998\sigma = 165$$ Solving the equations yields $\mu = 160$ g and $\sigma = 5$ g.
Specific behaviours
✓ correctly determines the Z-scores associated with $X = 150$ and $X = 165$ ✓ states the two simultaneous equations for μ and σ ✓ correctly solves for the mean and standard deviation

(b) Determine the probability that a tray of noodles contains no underweight servings. (3 marks)

Solution
Let the random variable Y denote the number of underweight servings in a tray. Then $$Y \sim \text{Bin}(20,\ 0.0228)$$ So $$P(Y = 0) = 0.6305$$ or Let the random variable Y denote the number of servings in a tray that are not underweight. Then $$Y \sim \text{Bin}(20,\ 0.9772)$$ So $$P(Y = 20) = 0.6305$$
Specific behaviours
✓ states that the distribution of trays that are underweight/not underweight is binomial ✓ states correct distribution parameters ✓ calculates correct probability

Question 12 (continued)

(c) Determine the margin of error of the confidence interval. (1 mark)

Solution
$E = \dfrac{0.1349 - 0.0651}{2}$ $= 0.0349$
Specific behaviours
✓ calculates the correct margin of error

(d) Determine the level of confidence that was used to calculate the confidence interval. (3 marks)

Solution
The sample proportion of underweight servings is $\hat{p} = \dfrac{0.1349 + 0.0651}{2} = 0.1$ Using the margin of error from part (c) $0.0349 = z\sqrt{\dfrac{0.1(1-0.1)}{200}}$ $z = 1.645$ Since $P(-1.645 \le Z \le 1.645) = 0.9$ it follows that it is a 90% confidence interval.
Specific behaviours
✓ calculates correct sample proportion   ✓ calculates correct critical value   ✓ determines correct confidence level

(e) On the basis of the above confidence interval, is the proportion of underweight servings of udon noodles different from what was claimed in the machine specifications? (2 marks)

Solution
The proportion of underweight medium udon noodle servings suggested by the machine specifications (p = 0.0228) is not within the above confidence interval (it is below the interval). Hence the sample provides sufficient evidence to conclude that the machine specification is not correct at the above confidence level.
Specific behaviours
✓ states that the machine specification proportion is not within the confidence interval   ✓ concludes that there is sufficient evidence at the above confidence level to conclude that the machine specification is incorrect

(f) All else remaining equal, state how the margin of error would change if

(i) the confidence level was decreased. (1 mark)

Solution
The margin of error would decrease
Specific behaviours
✓ states correct impact on the margin of error

(ii) the sample size was increased from 200 to 500. (1 mark)

Solution
The margin of error would decrease
Specific behaviours
✓ states correct impact on the margin of error

Question 13 **(9 marks)**

(a) (i) Determine the value of α. (2 marks)

Solution
From the diagram $$\tan\left(\frac{\alpha}{2}\right)=\frac{1}{10}$$ $$\Rightarrow \alpha = 2\tan^{-1}\left(\frac{1}{10}\right)$$ $$\approx 0.199$$
Specific behaviours
✓ writes correct expression for α ✓ obtains correct value for α

(ii) Hence show that the probability, p, of an apple rolling safely to the end of the ramp is $p = 0.063$ (rounded to three decimal places). (1 mark)

Solution
The probability of an apple rolling to the end of the ramp is given by $$p=\frac{\alpha}{\pi}$$ Hence $$p \approx \frac{0.199}{\pi}$$ $$=0.063$$
Specific behaviours
✓ demonstrates calculation of p as being $\frac{\alpha}{\pi}$

(b) Determine the probability that, of the 10 apples, four or more make it safely to the end of the ramp. (2 marks)

Solution
Based on the model assumptions, the number of apples that reach the end of the ramp X can be modelled as $$X \sim \text{Bin}(0.063, 10)$$ It follows that $$P(X \geq 4) = 0.0024$$
Specific behaviours
✓ states that the distribution is binomial and provides correct parameter values ✓ correctly calculates probability

(c) Using the sample of 200 apples, calculate a 99% confidence interval for the population proportion of apples that will roll safely to the end of the ramp. (2 marks)

Solution
The sample proportion is given by $$\hat{p} = \frac{63}{200} = 0.315$$ Hence the 99% confidence interval is given by $$99\%\ \text{CI} = \left(0.315 - 2.576\sqrt{\frac{0.315(1-0.315)}{200}},\ 0.315 + 2.576\sqrt{\frac{0.315(1-0.315)}{200}}\right) = (0.2304,\ 0.3996)$$
Specific behaviours
✓ calculates sample proportion correctly ✓ calculates confidence interval correctly

(d) What does the confidence interval from part (c) suggest about the validity of the model assumptions used to calculate the probability in part (a)(ii)? (2 marks)

Solution
The proportion calculated in part (a) is not within the 99% confidence interval (nowhere near close). This suggests that the model assumptions underpinning the calculation in part (a) are not valid.
Specific behaviours
✓ states that the probability from part (a) is not within the 99% CI ✓ concludes that the model assumptions are not valid

Question 14 **(11 marks)**

(a) Using calculus, show that the volume of water in the dam is given by

$$V(h) = 100h + \frac{80}{3}h^{\frac{3}{2}}.$$ (5 marks)

Solution
Determine an expression for x in terms of h by solving $$h = \frac{(x-5)^2}{4}$$ $$\Rightarrow 4h = (x-5)^2$$ $$\Rightarrow x = 5 + 2\sqrt{h}$$ Hence the volume is given by $$V(h) = 20\left(5h + \int_5^{5+2\sqrt{h}} h - \frac{(x-5)^2}{4}\,dx\right)$$ $$= 20\left(5h + \left[hx - \frac{(x-5)^3}{12}\right]_5^{5+2\sqrt{h}}\right)$$ $$= 20\left(5h + \left(h\left(5+2\sqrt{h}\right) - \frac{\left(2\sqrt{h}\right)^3}{12}\right) - 5h\right)$$ $$= 20\left(5h + \frac{4}{3}h^{\frac{3}{2}}\right)$$ $$= 100h + \frac{80}{3}h^{\frac{3}{2}}$$
Specific behaviours
✓ determines an expression for the upper bound of the volume integral ✓ states a correct integral expression for the volume or cross-sectional area of water ✓ states correct antiderivative of integrand ✓ correctly applies fundamental theorem of calculus by substituting integration bounds ✓ simplifies to give desired result

(b) Use the increments formula to estimate the change in water volume if the water level rises from 6 m to 6.1 m. (3 marks)

Solution
The derivative of V with respect to h is $$\frac{dV}{dh} = 100 + 40\sqrt{h}$$ The change in h is $$\delta h = 6.1 - 6 = 0.1$$ Hence the change in V is approximately $$\delta V \approx \frac{dV}{dh}\delta h = \left(100 + 40\sqrt{h}\right) \times 0.1 = 10 + 4\sqrt{h}$$ When $h = 6$ we have $$\delta V \approx 10 + 4\sqrt{6} \approx 19.80 \text{ m}^3$$
Specific behaviours
✓ determines correct expression for the derivative of V with respect to h ✓ determine the increment in h ✓ obtains correct estimate for the change in V (exact or decimal)

(c) Assuming that there are no other sources of water and no losses, determine the probability that the dam will reach full capacity (i.e. depth of 10 m) during winter. (3 marks)

Solution
The full capacity of the dam is $$V(10) = 100(10) + \frac{80}{3}(10)^{\frac{3}{2}} \approx 1843.27 \text{ m}^3$$ Hence the dam will reach capacity if $$V_R \geq V(10) - 1000 = 1843.27 - 1000 = 843.27 \text{ m}^3$$ Since $V_R \sim N\left(600,\ 200^2\right)$ it follows that $$P\left(V_R \geq 843.27\right) \approx 0.1119$$
Specific behaviours
✓ determines the correct volume $V(10)$ ✓ determines that we need $V_R \geq 843.27$ ✓ obtains correct probability

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303 Sevenoaks Street
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