2023/74321

Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS SPECIALIST

Calculator-free

ATAR course examination 2023

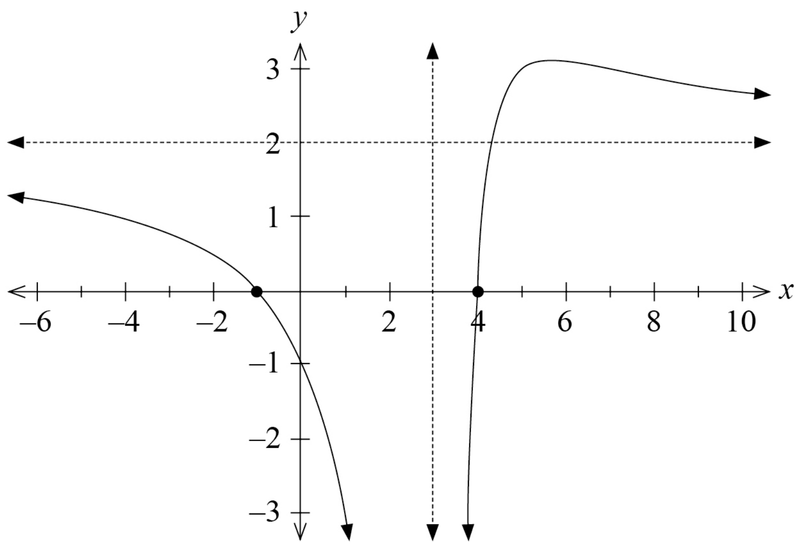
Marking key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

Section One: Calculator-free **35% (48 Marks)**

Question 1 **(4 marks)**

The graph of the function $f(x) = \dfrac{k(x+a)(x-b)}{(x-c)^2}$ is shown below. The constants $a,\ b,\ c$ and k are positive.



Complete the table below by determining the values for $a,\ b,\ c$ and k.

a	b	c	k
1	**4**	**3**	**2**

Solution
The x intercepts are $x=-1,\ x=4 \quad \therefore \quad a=1,\ b=4$ Vertical asymptote is $x=3 \quad \therefore \quad c=3$ Horizontal asymptote is $y=2 \quad \therefore \quad k=2$
Specific behaviours
✓ states the value for a and b correctly ✓ states the value for c correctly ✓ states the value for k correctly ✓ provides justification for at least one of the values for $a,\ b,\ c,\ k$

Question 2 **(5 marks)**

$P(z) = z^5 + az^4 + bz^3 + cz^2 + dz + 14$ is a fifth order polynomial with real coefficients. It is known that $P(z) = (z - z_0)Q(z)$ where z_0 is real and $Q(z)$ is a fourth order polynomial. Two roots of $P(z)$ are $z_1 = 1 + i$ and $z_2 = 2 + \sqrt{3}i$.

(a) Determine $Q(z)$ in expanded form. (3 marks)

Solution
If two roots are $1+i$ and $2+\sqrt{3}i$, then so are the conjugates $1-i$ and $2-\sqrt{3}i$. $\therefore\ Q(z) = (z-(1+i))(z-(1-i))\left(z-\left(2+\sqrt{3}i\right)\right)\left(z-\left(2-\sqrt{3}i\right)\right)$ $= ((z-1)-i)((z-1)+i)\left((z-2)-\sqrt{3}i\right)\left((z-2)+\sqrt{3}i\right)$ $= \left((z-1)^2+1\right)\left((z-2)^2+3\right)$ $= (z^2-2z+2)(z^2-4z+7) \quad \ldots (1)$ $= z^4-6z^3+17z^2-22z+14 \quad \ldots (2)$
Specific behaviours
✓ states that are $1-i$ and $2-\sqrt{3}i$ are also roots of $P(z)$ ✓ expresses $Q(z)$ as a product of four linear factors correctly ✓ determines $Q(z)$ as either expression (1) or (2)

(b) Determine the values of the coefficients $a,\ b,\ c$ and d. (2 marks)

Solution
Since $P(z) = z^5+az^4+bz^3+cz^2+dz+14$ $= (z-z_0)(z^4-6z^3+17z^2-22z+14)$ Then the constant term $14 = (-z_0)(14) \quad \therefore\ z_0 = -1$ $\therefore\ P(z) = (z+1)(z^4-6z^3+17z^2-22z+14)$ $= z^5-5z^4+11z^3-5z^2-8z+14$ Hence $a=-5,\ b=11,\ c=-5,\ d=-8$
Specific behaviours
✓ determines the value for z_0 ✓ states the values for $a,\ b,\ c$ and d correctly

Question 3 **(5 marks)**

Using the substitution $x = 119u + 1$, evaluate exactly $\int_1^{120} \left(2 + 4\left(\frac{x+118}{119} \right)^3 \right) dx$.

Solution
Using $x = 119u + 1 \quad \therefore \quad dx = 119\,du$
$x + 118 = 119u + 119$
$\therefore \quad \frac{x+118}{119} = \frac{119u+119}{119} = u + 1$

x	1	120
u	0	1

$$\int_1^{120} \left(2 + 4\left(\frac{x+118}{119} \right)^3 \right) dx = \int_0^1 \left(2 + 4(u+1)^3 \right) .119\,du$$

$$= 119 \left[2u + (u+1)^4 \right]_0^1$$

$$= 119 \left[\left(2 + 2^4\right) - \left(0 + 1^4\right) \right]$$

$$= 119 \left[18 - 1 \right]$$

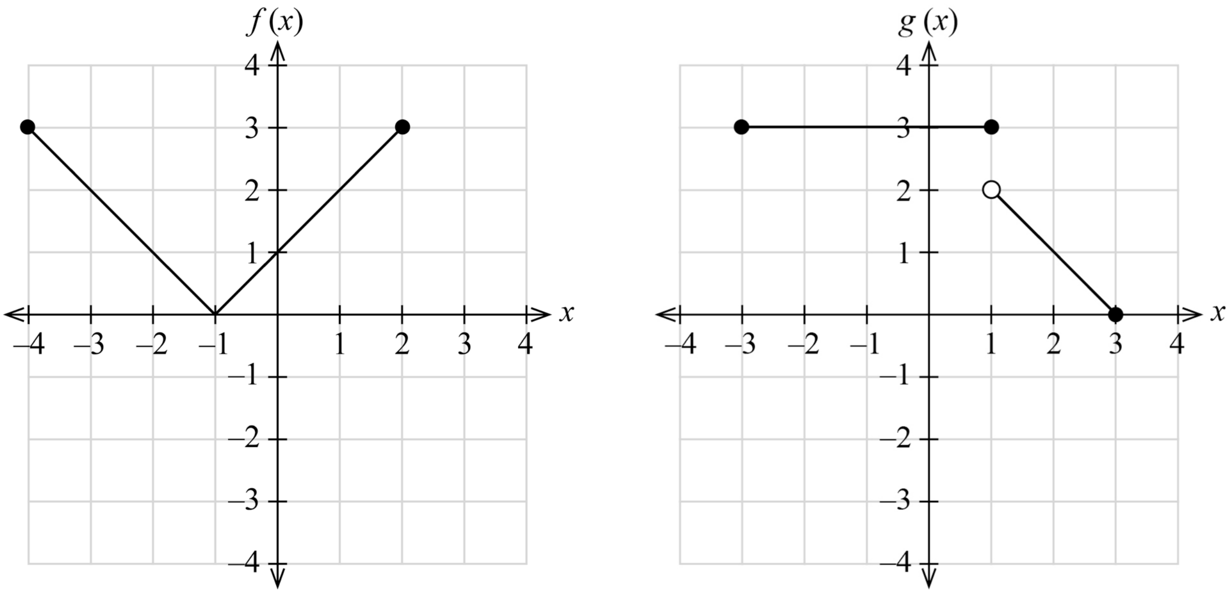
$$= 119 \times 17$$

$$= 2023$$

Specific behaviours
✓ writes dx correctly in terms of du
✓ changes the limits correctly
✓ simplifies the integrand correctly in terms of the variable u
✓ anti-differentiates correctly
✓ evaluates the definite integral correctly

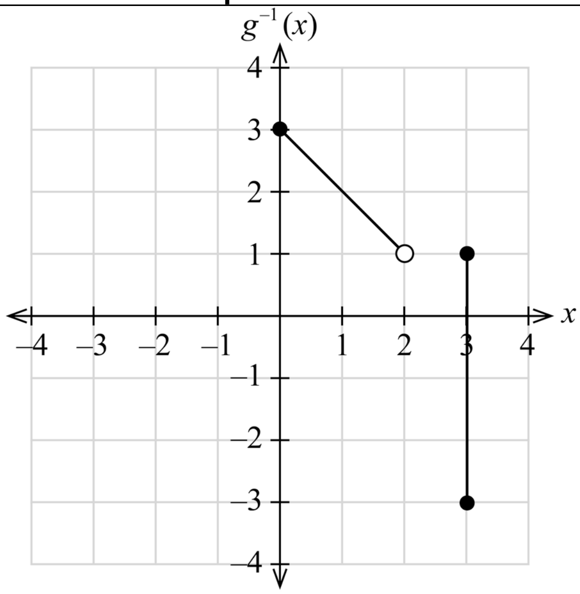
Question 4 **(11 marks)**

The graphs of functions $f(x)$ and $g(x)$ are shown.



(a) Sketch the graph of $y = g^{-1}(x)$ on the axes below. (2 marks)

Solution
Inverse of $g(x)$ does not exist as $g(x)$ is not a one-to-one function. (Graph cannot be sketched).
Specific behaviours
✓✓ identifies inverse of $g(x)$ does not exist or identifies graph cannot be sketched

Accepted solution

Specific behaviours
✓ indicates $x = 3$ for $-3 \le y \le 1$ ✓ indicates $y = 3 - x$ for $0 \le x < 2$

Question 4 (continued)

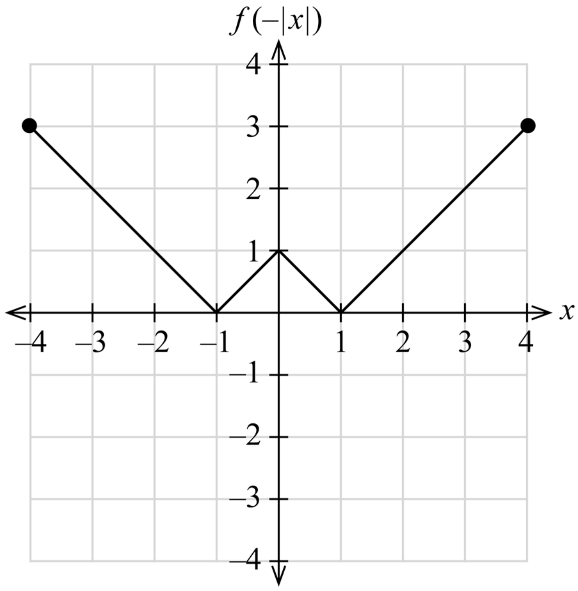
(b) State the value for $g\left(f^{-1}(0)\right)$. (2 marks)

Solution
Let $f^{-1}(0)=x \quad \therefore \; f(x)=0$ From the graph of $y=f(x)$ hence $x=-1$. $g\left(f^{-1}(0)\right) \;=\; g(-1) \;=\; 3$
Specific behaviours
✓ states that $f^{-1}(0)=-1$ ✓ evaluates $g\left(f^{-1}(0)\right)$ correctly

(c) Determine the set of values of x such that $f\left(g(x)\right)$ is defined. (2 marks)

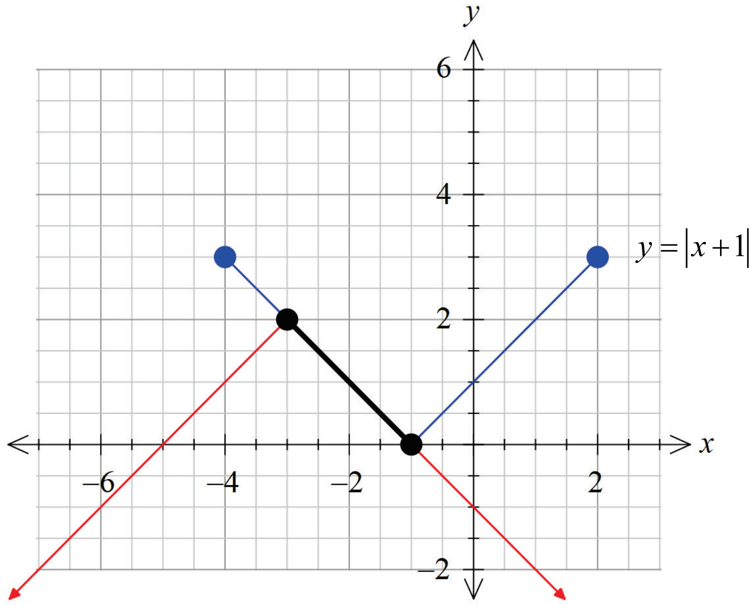
Solution
For $f\left(g(x)\right)$ to be defined then $R_g \subseteq D_f$ i.e. the range of g must be part of the domain of f. This will occur when $g(x)\le 2$ i.e. $1<x\le 3$.
Specific behaviours
✓ states that $R_g \subseteq D_f$ ✓ states the correct set of values for x

(d) Sketch the graph of $y=f\left(-\lvert x\rvert\right)$ on the axes below. (2 marks)



Solution
Shown above.
Specific behaviours
✓ indicates symmetry about $x=0$ ✓ indicates the correct set of points for $-4\le x\le 4$

(e) The equation $|x+1| = k - |x+a|$ has an infinite number of solutions, with the solution set being $-3 \le x \le -1$. Determine the values of the constants a and k. (3 marks)

Solution
Consider the graph of $y=\|x+1\|$ and $y=k-\|x+a\|$ so that they intersect only when $-3 \le x \le -1$.  This intersection will occur when we consider $y=2-\|x+3\|$. Hence $k=2$ and $a=3$.
Specific behaviours
✓ states the correct value for k ✓ states the correct value for a ✓ provides appropriate justification (considers the graphs of absolute value functions that yields an intersection only for $-3 \le x \le -1$)
Accept other relevant answers.

Question 5 **(5 marks)**

Consider two planes given by their Cartesian equations:

$$x-3y+3z=9$$
$$2x+y-z=4$$

(a) Explain why these planes are not parallel. (1 mark)

Solution
The normal vectors for each plane $\begin{pmatrix}1\\-3\\3\end{pmatrix}$ and $\begin{pmatrix}2\\1\\-1\end{pmatrix}$ are not scalar multiples of each other. Hence the planes cannot be parallel to each other.
Specific behaviours
✓ explains that the normal vectors are not multiples of each other

(b) State the geometric interpretation of the solution in the above simultaneous equations. (1 mark)

Solution
the two planes will intersect in a line in space
Specific behaviours
✓ states that the planes intersect in a line

(c) Determine the vector equation for the intersection of these two planes. (3 marks)

Solution
$x-3y+3z=9 \quad \ldots (1)$ $2x+y-z=4 \quad \ldots (2)$ Consider $(1) + 3\times(2)$: $7x+0y+0z=9+12$ i.e. $7x=21$ $\therefore\ x=3$ Substituting $x=3$ into (1): $3-3y+3z=9$ i.e. $z=y+2$ where $y\in\mathbb{R}$ i.e. there are infinitely many ordered triples for x, y, z. Hence the intersection of the two planes is a line in space. Vector equation for this line: $\underset{\sim}{r} = \begin{pmatrix}3\\\lambda\\\lambda+2\end{pmatrix} = \begin{pmatrix}3\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\1\end{pmatrix}$
Specific behaviours
✓ eliminates a variable correctly from the pair of equations ✓ obtains the relationship $z=y+2$ ✓ forms the vector equation of the line using a parameter correctly

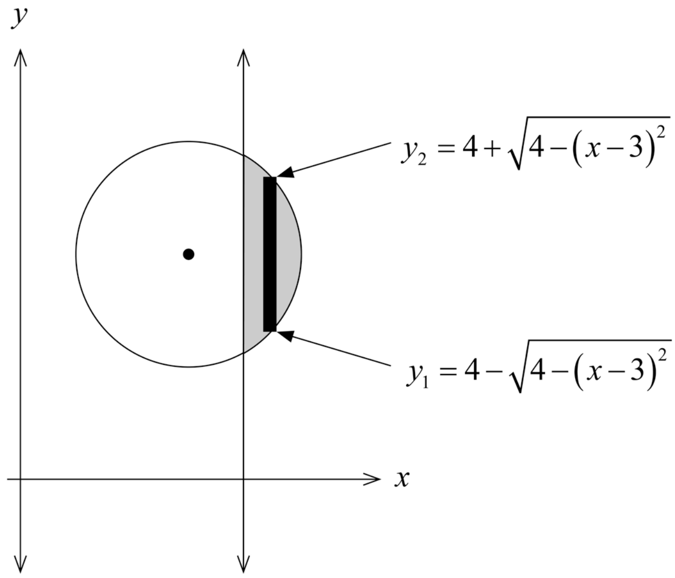
Question 6 **(5 marks)**

Solve the complex equation $z^4 = 2 - 2\sqrt{3}\,i$ giving solutions in the form $rcis\theta$ where $-\pi < \theta \le \pi$.

Solution
$\left\|z^4\right\| = \sqrt{2^2 + \left(2\sqrt{3}\right)^2} = \sqrt{4+12} = 4 \quad Arg\left(z^4\right) = \tan^{-1}\left(\frac{-2\sqrt{3}}{2}\right) = \tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$ $\therefore$ Solve $z^4 = 4cis\left(-\frac{\pi}{3}\right)$ $\therefore\ z = 4^{\frac{1}{4}}cis\left(\frac{-\pi}{12} + k\left(\frac{\pi}{2}\right)\right) \quad k = 0,1,2,3 \quad \ldots (1)$ Roots are: $z_0 = \sqrt{2}\,cis\left(-\frac{\pi}{12}\right)$ $z_1 = \sqrt{2}\,cis\left(\frac{5\pi}{12}\right)$ $z_2 = \sqrt{2}\,cis\left(\frac{11\pi}{12}\right)$ $z_3 = \sqrt{2}\,cis\left(-\frac{7\pi}{12}\right)$ Note: $z_3 = \sqrt{2}\,cis\left(\frac{17\pi}{12}\right)$ does not satisfy $-\pi < \theta \le \pi$.
Specific behaviours
✓ states the value for $\left\|z^4\right\|$ correctly ✓ states the value for $Arg\left(z^4\right)$ correctly ✓ states the principal solution $z_0 = \sqrt{2}\,cis\left(-\frac{\pi}{12}\right)$ ✓ indicates a separation of $\frac{\pi}{2}$ between solution arguments ✓ states all solutions correctly using the condition $-\pi < \theta \le \pi$

Question 7 **(9 marks)**

The shaded region is bounded by the curve $(x-3)^2+(y-4)^2=4$ and the line $x=4$.



(a) Show that the area of this region is given by the definite integral $\int_4^a 2\sqrt{4-(x-3)^2}\,dx$.

State the value for a. (3 marks)

Solution
Circle radius is 2 units so $a=3+2=5$ From equation of circle $(y-4)^2 = 4-(x-3)^2 \quad \therefore \quad y-4 = \pm\sqrt{4-(x-3)^2}$ $y = 4 \pm \sqrt{4-(x-3)^2}$ Area $= \int_4^5 (y_2-y_1)\,dx = \int_4^5 \left(4+\sqrt{4-(x-3)^2}\right)-\left(4-\sqrt{4-(x-3)^2}\right)dx$ $= \int_4^5 2\sqrt{4-(x-3)^2}\,dx$
Specific behaviours
✓ states the value for a correctly ✓ obtains $y = 4 \pm \sqrt{4-(x-3)^2}$ correctly from the equation of the circle ✓ forms a difference of y values to obtain the integrand $2\sqrt{4-(x-3)^2}$

(b) By using the substitution $x-3=2\sin\theta$, determine the exact value for the area of the shaded region. (6 marks)

Solution

x	4	5
θ	$\frac{\pi}{6}$	$\frac{\pi}{2}$

$$x = 3 + 2\sin\theta$$

$$\frac{dx}{d\theta} = 2\cos\theta \qquad \therefore \quad dx = 2\cos\theta\, d\theta$$

$$\int_4^5 2\sqrt{4-(x-3)^2}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\sqrt{4-4\sin^2\theta}\,.\,2\cos\theta\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\sqrt{4\cos^2\theta}\,.\,2\cos\theta\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8\cos^2\theta\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(4\cos 2\theta + 4\right) d\theta = \left[2\sin 2\theta + 4\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left[2(0) + 2\pi\right] - \left[2\left(\frac{\sqrt{3}}{2}\right) + \frac{2\pi}{3}\right]$$

$$= \frac{4\pi}{3} - \sqrt{3} \quad \text{square units}$$

Specific behaviours

✓ changes the limits correctly
✓ obtains dx correctly in terms of $d\theta$
✓ simplifies the integrand in terms of θ correctly (Pythagorean identity)
✓ uses the cosine double angle identity correctly
✓ anti-differentiates the trigonometric function correctly
✓ evaluates the definite integral correctly

Question 8 **(4 marks)**

In the following simultaneous equations, a and b are real numbers.

$$a^3 = 3ab^2 + 14$$
$$b^3 = 3a^2b + 2\sqrt{5}$$

In order to determine the value of $a^2 + b^2$ from these equations, it is useful to consider the complex expansion for $(a+bi)^3$. Hence, or otherwise, determine the exact value of $a^2 + b^2$.

Solution
$(a+bi)^3 = a^3 + 3a^2(bi) + 3a(bi)^2 + (bi)^3$ $= a^3 + (3a^2b)i - 3ab^2 - (b^3)i$ $= (a^3 - 3ab^2) + (3a^2b - b^3)i$ From equation (1) we have: $a^3 - 3ab^2 = 14$ From equation (2) : $3a^2b - b^3 = -2\sqrt{5}$ Hence $(a+bi)^3 = 14 - 2\sqrt{5}\,i$. $\therefore\ \left\|(a+bi)^3\right\| = \sqrt{14^2 + (2\sqrt{5})^2} = \sqrt{196+20} = \sqrt{216}$ $\therefore\ \left\|(a+bi)\right\|^3 = \sqrt{216}$ since $\|z^3\| = \|z\|^3$ $\therefore\ \|a+bi\| = (216)^{\frac{1}{6}}$ $\therefore\ \|a+bi\|^2 = (216)^{\frac{1}{3}}$ i.e. $a^2 + b^2 = \sqrt[3]{216} = 6$ Note: accept $\sqrt[3]{216}$ as the final answer.
Specific behaviours
✓ obtains $(a+bi)^3$ correctly as $(a^3 - 3ab^2) + (3a^2b - b^3)i$ or its equivalent ✓ deduces $(a+bi)^3 = 14 - 2\sqrt{5}\,i$ ✓ obtains the value for $\|a+bi\| = (216)^{\frac{1}{6}}$ or its equivalent ✓ deduces the value of $a^2 + b^2$

Published by the School Curriculum and Standards Authority of Western Australia
303 Sevenoaks Street
CANNINGTON WA 6107