Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS METHODS

Calculator-free

ATAR course examination 2019

Marking key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2019/63006

Section One: Calculator-free **35% (52 Marks)**

Question 1 **(6 marks)**

Consider the derivative function $f'(x) = xe^{x^2}$.

(a) Determine $f''(1)$. (2 marks)

Solution
$f''(x) = x(2xe^{x^2}) + e^{x^2}$ $f''(1) = 3e$
Specific behaviours
✓ uses the chain rule to correctly differentiate $f'(x)$ ✓ evaluates at $x = 1$

(b) Explain the meaning of your answer to part (a). (1 mark)

Solution
$f''(1)$ is the rate of change of the derivative function when $x = 1$
Specific behaviours
✓ states it is the rate of change of the derivative AND includes when $x = 1$

(c) Determine the expression for $y = f(x)$, given that it intersects the y-axis at the point (0,2).

(3 marks)

Solution
$\int xe^{x^2}dx = \frac{e^{x^2}}{2} + C$ $2 = \frac{e^0}{2} + C$ $C = \frac{3}{2}$ $\therefore f(x) = \frac{e^{x^2}}{2} + \frac{3}{2}$
Specific behaviours
✓ correctly integrates $f'(x)$ ✓ substitutes (0,2) into an expression involving C ✓ determines C and states the final expression for $y = f(x)$

Question 2 **(6 marks)**

The values of the functions $g(x)$ and $h(x)$, and their derivatives $g'(x)$ and $h'(x)$ are provided in the table below for $x = 1$, $x = 2$ and $x = 3$.

	$x = 1$	$x = 2$	$x = 3$
$g(x)$	3	5	–3
$h(x)$	2	–2	6
$g'(x)$	−4	1	4
$h'(x)$	0	–6	–5

(a) Evaluate the derivative of $\dfrac{g(x)}{h(x)}$ at $x = 3$. (2 marks)

Solution
$\left(\dfrac{g}{h}\right)'(3) = \dfrac{g'(3)h(3) - g(3)h'(3)}{h(3)^2}$ $= \dfrac{4(6) - (-3)(-5)}{6^2}$ $= \dfrac{1}{4}$
Specific behaviours
✓ expresses the derivative using the quotient rule ✓ evaluates the derivative

(b) Evaluate the derivative of $h(g(x))$ at $x = 1$. (2 marks)

Solution
$h(g(1))' = h'(g(1))g'(1)$ $= h'(3)(-4)$ $= (-5)(-4)$ $= 20$
Specific behaviours
✓ expresses the derivative using the chain rule ✓ evaluates the derivative

(c) If $h''(1) = -1$, describe with justification, what the graph of $h(x)$ looks like at this point. (2 marks)

Solution
Since $h'(1) = 0$ there is a stationary point at $x = 1$ Since 2nd derivative is negative the point is a maximum
Specific behaviours
✓ justifies stationary point ✓ determines point is a maximum

Question 3 **(7 marks)**

Waiting times for patients at a hospital emergency department can be up to four hours. The associated probability density function is shown below.

(a) What is the probability a patient will wait less than one hour? (3 marks)

Solution
For: $0 \le t \le 1.5$ $f(t) = \frac{0.5}{1.5}t = \frac{t}{3}$ $P(T \le 1) = \int_0^1 \frac{t}{3}\,dt = \left[\frac{t^2}{6}\right]_0^1 = \frac{1}{6}$
Specific behaviours
✓ determines equation for $f(t)$ ✓ writes a correct statement for probability involving calculus   ✓ evaluates integral to determine probability

OR

Alternate Solution
Required probability is the area of the triangle that has base 1 unit   The height of the triangle is $\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$ $P(T \le 1) = \frac{1}{2} \times 1 \times \frac{1}{3} = \frac{1}{6}$
Specific behaviours
✓ recognises that the probability is the area of triangle with base length 1 unit   ✓ determines the height of the triangle   ✓ correctly calculates the area

(b) What is the probability a patient will wait between one hour and three hours? (4 marks)

Solution

$P(1 \le T \le 3) = 1 - P(0 \le T \le 1) - P(3 \le T \le 4)$

$P(0 \le T \le 1) = \frac{1}{6}$

For: $1.5 \le t \le 4$

$f(t) = -\frac{0.5}{2.5}t + c = -\frac{t}{5} + c$

$f(4) = 0$

$0 = -\frac{4}{5} + c \Rightarrow \frac{4}{5}$

$f(t) = -\frac{t}{5} + \frac{4}{5}$

$P(3 \le T \le 4) = \frac{1}{5}\int_3^4 (-t+4)dt = \frac{1}{5}\left[-\frac{t^2}{2} + 4t\right]_3^4$

$= \frac{1}{5}\left[-8 + 16 + \frac{9}{2} - 12\right]$

$= \frac{1}{10}$

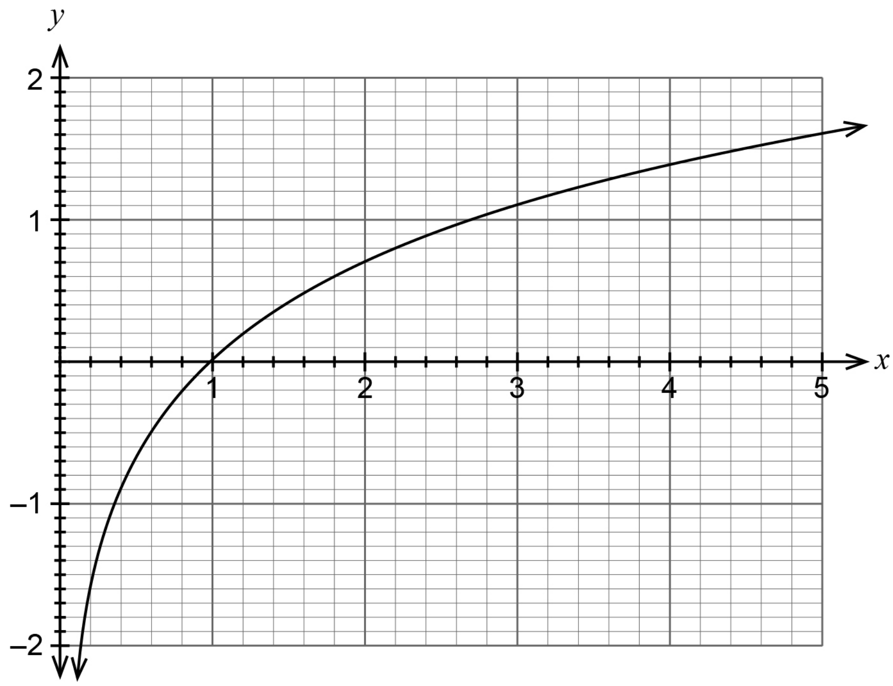
$P(1 \le T \le 3) = 1 - \frac{1}{10} - \frac{1}{6} = \frac{22}{30} = \frac{11}{15}$

Specific behaviours

✓ determines the equation for $f(t)$ when $1.5 \le t \le 4$
✓ writes a correct statement for probability
✓ calculates $P(3 \le T \le 4)$ correctly
✓ calculates $P(1 \le T \le 3)$ correctly

Question 4 **(6 marks)**

Consider the graph of $y = \ln(x)$ shown below.



(a) Use the graph to estimate the value of p in each of the following.

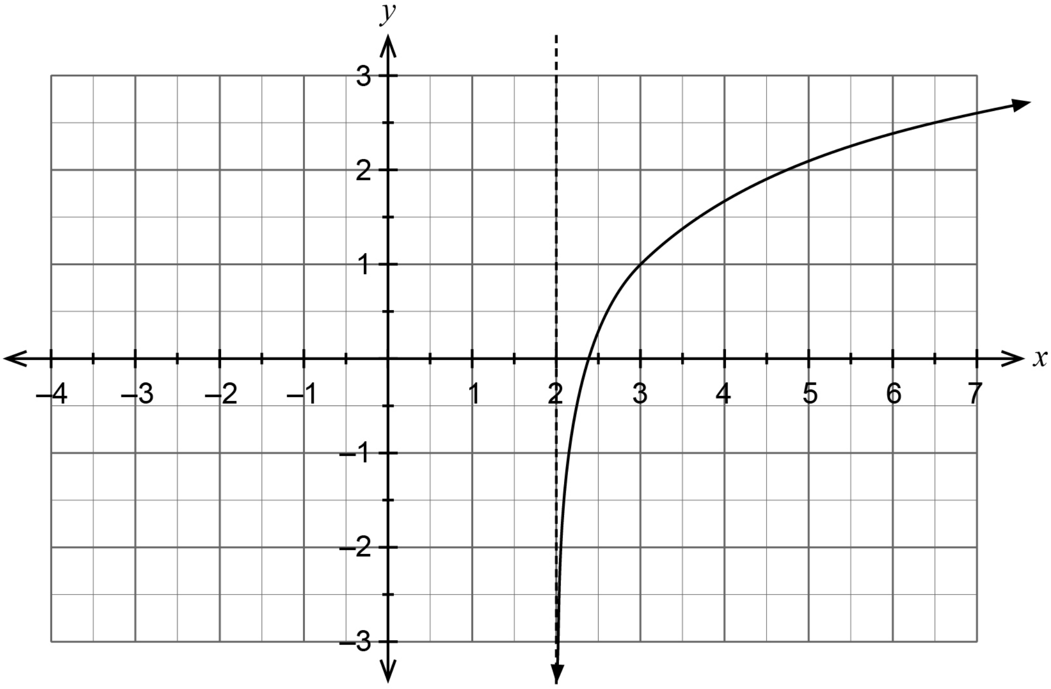
(i) $1.4 = \ln(p)$ (1 mark)

Solution
$p = 4$
Specific behaviours
✓ states the correct value of p

(ii) $e^{p+1} - 3 = 0$ (2 marks)

Solution
$e^{p+1} = 3$ $p + 1 = \ln(3)$ $p + 1 = 1.1$ $\therefore p = 0.1$
Specific behaviours
✓ rearranges to form a logarithmic equation ✓ states the correct value of p

(b) On the axes below, sketch the graph of $y = \ln(x-2)+1$. (3 marks)



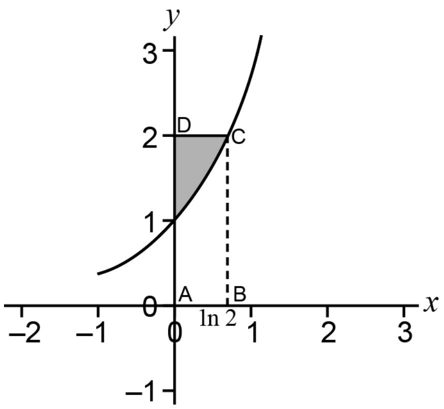
Solution
Specific behaviours
✓ draws asymptote at $x=2$ ✓ the sketch passes through the point (3,1) ✓ the sketch has the correct shape and has a y-coordinate between 2.5 and 3 when $x=7$

Question 5 **(8 marks)**

(a) Determine the area bound by the graph of $f(x) = e^x$ and the x-axis between $x = 0$ and $x = \ln 2$. (3 marks)

Solution
First we obtain the area under the graph of $f(x)$between $x = 0$ and $x = \ln 2$. This is given by $$A = \int_0^{ln2} e^x dx = e^x\|_0^{ln2} = 2 - 1 = 1.$$
Specific behaviours
✓ writes down the correct integral ✓ integrates correctly ✓ simplifies to obtain final answer

(b) Hence, determine the area bound by the graph of $f(x) = e^x$, the line $y = 2$ and the y-axis. (2 marks)

Solution
This is given by the area shown below.

That is, $$Area = 2\ln 2 - 1$$
Specific behaviours
✓ correctly defines the area ✓ calculates the area correctly

(c) Determine the area bounded by the graph of $f(x) = e^x$, the line $y = a$ and the y-axis, where a is a positive constant. (3 marks)

Solution
$$\int_0^{lna} e^x dx = e^x\|_0^{lna} = a - 1$$ $$A = \ln(a) \times a - (a-1)$$ $$= a\ln(a) - a + 1$$
Specific behaviours
✓ writes down the correct integral ✓ integrates correctly and simplifies to obtain $a–1$ ✓ determines the correct expression for area

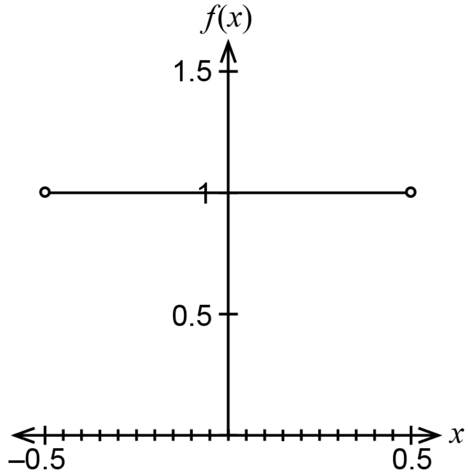
Question 6 **(10 marks)**

The error X in digitising a communication signal has a uniform distribution with probability density function given by

$$f(x) = \begin{cases} 1, & -0.5 < x < 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch the graph of $f(x)$. (2 marks)

Solution



Specific behaviours

✓ draws horizontal line at y=1
✓ indicates a domain of $-0.5 < x < 0.5$

(b) What is the probability that the error is at least 0.35? (1 mark)

Solution
$P(X > 0.35) = Area = 0.15$
Specific behaviours
✓ computes the correct probability

(c) If the error is negative, what is the probability that it is less than –0.35? (2 marks)

Solution
$P(X < -0.35 \mid X < 0) = \dfrac{P(X < -0.35 \cap X < 0)}{P(X < 0)} = \dfrac{P(X < -0.35)}{P(X < 0)} = \dfrac{0.15}{0.5} = 0.3$
Specific behaviours
✓ writes the correct conditional probability statement ✓ computes the probability correctly

(d) An engineer is more interested in the square of the error. What is the probability that the square of the error is less than 0.09? (2 marks)

Solution
$P(X^2 < 0.09) = P(-0.3 < X < 0.3) = 0.6$
Specific behaviours
✓ correctly expresses the required probability in terms of X ✓ computes the probability correctly

Question 6 (continued)

(e) Calculate the variance of the error. (3 marks)

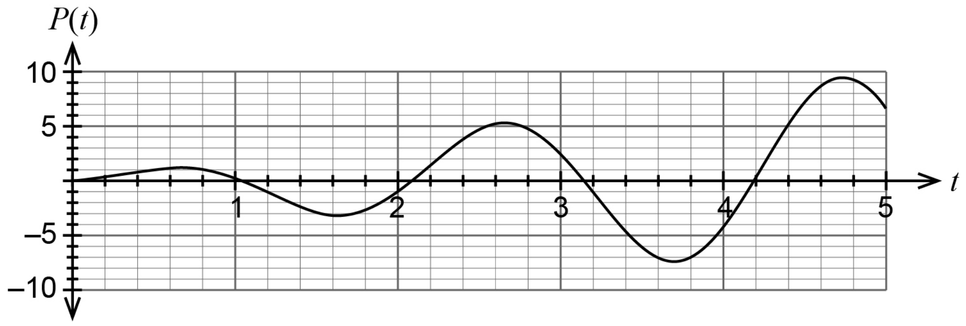
Solution
$$E(X) = \int_{-0.5}^{0.5} x\,dx = 0$$ So $$Var(X) = \int_{-0.5}^{0.5} (x-0)^2(1)dx = \left.\frac{x^3}{3}\right\|_{-0.5}^{0.5} = \frac{0.125 + 0.125}{3} = \frac{1}{12}$$
Specific behaviours
✓ computes mean correctly ✓ states an integral for the variance ✓ evaluates the integral to determine variance correctly

Question 7 **(9 marks)**

A company's profit, in millions of dollars, over a five-year period can be modelled by the function:

$$P(t) = 2t\sin(3t) \quad 0 \le t \le 5 \text{ where } t \text{ is measured in years.}$$

The graph of $P(t)$ is shown below.



(a) Differentiate $P(t)$ to determine the marginal profit function, $P'(t)$. (2 marks)

Solution
$P'(t) = 2\sin(3t) + 6t\cos(3t) \quad \$/year$
Specific behaviours
✓ uses the product rule ✓ determines correct derivative

(b) Calculate the rate of change of the marginal profit function when $t = \dfrac{\pi}{18}$ years. (4 marks)

Solution
$P'(t) = 2\sin(3t) + 6t\cos(3t)$ $P''(t) = 6\cos(3t) + 6\cos(3t) - 18t\sin(3t)$ $= 12\cos(3t) - 18t\sin(3t)$ $P''\left(\dfrac{\pi}{18}\right) = 12\cos\left(\dfrac{\pi}{6}\right) - \pi\sin\left(\dfrac{\pi}{6}\right)$ $= 6\sqrt{3} - \dfrac{\pi}{2} \quad \$/year^2$
Specific behaviours
✓ uses product rule ✓ determines correct expression for the second derivative ✓ substitutes $\dfrac{\pi}{18}$ into second derivative expression ✓ calculates the exact rate of change

Question 7 (continued)

(c) Use the increments formula at $t = \frac{7\pi}{6}$ to estimate the change in profit for a one month change in time. (3 marks)

Solution
$P'\left(\frac{7\pi}{6}\right) = 2\sin\left(\frac{7\pi}{2}\right) + 6\left(\frac{7\pi}{6}\right)\cos\left(\frac{7\pi}{2}\right)$ $= -2$ $\delta P \approx \frac{dP}{dt} \times \delta t$ $\approx -2 \times \frac{1}{12}$ $\approx -\frac{1}{6}$ The approximate change in profit is $-\frac{1}{6}$ million dollars. [$\frac{1}{6}$ million dollar loss]
Specific behaviours
✓ calculates the correct value of P' when $t = \frac{7\pi}{6}$ ✓ states an appropriate approximation for the change in profit using the increments formula ✓ substitutes and evaluates the change including units

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