2024/81907

Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS SPECIALIST

Calculator-free

ATAR course examination 2024

Marking key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

Section One: Calculator-free **35% (47 Marks)**

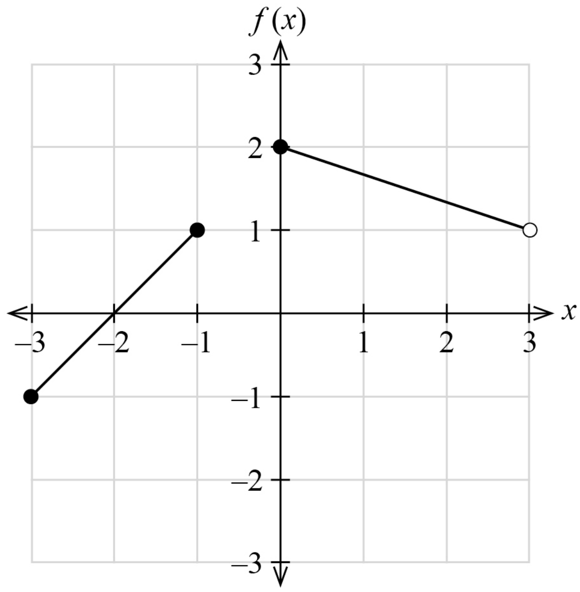
Question 1 **(3 marks)**

The complex number $z = r\,cis\,\theta \;=\; 3 + bi$, where $\tan\theta = \sqrt{2}$. Determine the exact values for r and b.

Solution
Given $z = r\,cis\,\theta$ where $\tan\theta = \sqrt{2}$ $\quad\therefore\quad \frac{b}{3} = \sqrt{2}$ $\quad$ i.e. $b = 3\sqrt{2}$ $r^2 = 3^2 + b^2$ $r^2 = 9 + \left(3\sqrt{2}\right)^2 = 9 + 9(2) \qquad \therefore\; r^2 = 27$ i.e. $r = \sqrt{27} = 3\sqrt{3}$
Specific behaviours
✓ states $b = 3\sqrt{2}$ ✓ states $r = \sqrt{27}$ or $3\sqrt{3}$ ✓ justifies the determination of both b and r

Question 2 **(5 marks)**

The graph of $y = f(x)$ is shown below.



(a) State the domain for $g(x) = \sqrt{f(x) - 1}$. Justify your answer. (3 marks)

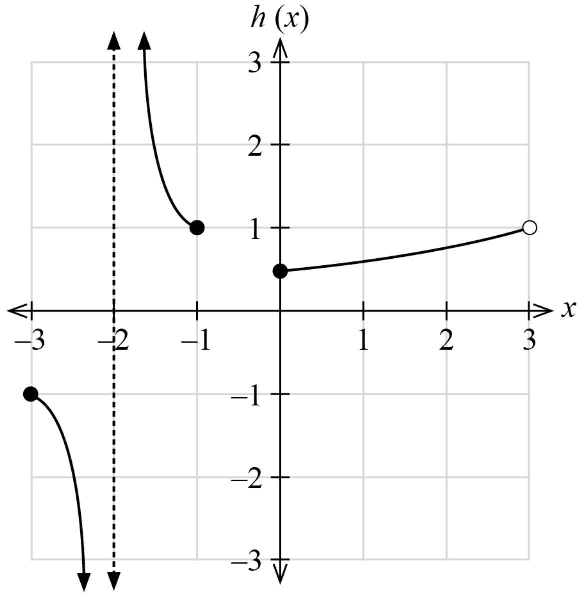
Solution
For $y = \sqrt{f(x) - 1}$ to be defined we require $f(x) - 1 \geq 0$. i.e. $f(x) \geq 1$ This occurs when $x = -1, \quad 0 \leq x < 3$.
Specific behaviours
✓ states $x = -1$ ✓ states $0 \leq x < 3$ ✓ justifies by referring to the requirement that $f(x) \geq 1$

Question 2 (continued)

(b) State the range for function $h(x) = \dfrac{1}{f(x)}$. (2 marks)

Solution

Consider the sketch of $h(x) = \dfrac{1}{f(x)}$.



$\therefore \quad R_h = \{ y \mid y \le -1, \; y \ge \frac{1}{2} \}$

Specific behaviours

✓ states $y \le -1$

✓ states $y \ge \dfrac{1}{2}$

Question 3 **(4 marks)**

Using the substitution $u = 1 - x$, evaluate exactly the definite integral $\int_0^1 15x\sqrt{1-x}\,dx$.

Solution

Let $u = 1 - x \quad \therefore \quad \frac{du}{dx} = -1 \quad$ i.e. $dx = -du$

x	0	1
u	1	0

$$\int_0^1 15x\sqrt{1-x}\,dx = \int_1^0 15(1-u)\times\sqrt{u}\times(-du)$$

$$= \int_1^0 -15(1-u)u^{\frac{1}{2}}du$$

$$= \int_0^1 15\left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du$$

$$= 15\left[\frac{2u^{\frac{3}{2}}}{3} - \frac{2u^{\frac{5}{2}}}{5}\right]_0^1 = 15\left(\frac{2}{3} - \frac{2}{5}\right)$$

$$= 15\times\frac{4}{15} = 4$$

Specific behaviours

- ✓ changes the limits correctly
- ✓ uses the substitution to express the integrand correctly in terms of u
- ✓ anti-differentiates correctly
- ✓ evaluates correctly

Question 4 **(6 marks)**

(a) Given that $\dfrac{a}{x+1} + \dfrac{b}{(x+1)^2} = \dfrac{5x+3}{(x+1)^2}$, determine the values for a and b. (2 marks)

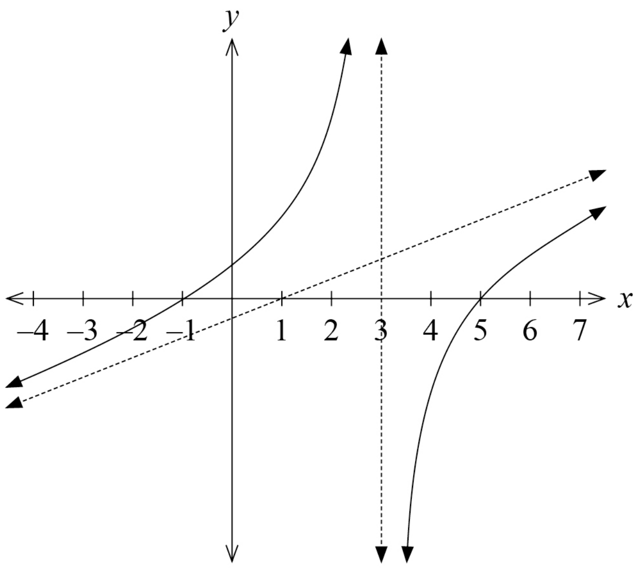
Solution
$\dfrac{a}{x+1} + \dfrac{b}{(x+1)^2} = \dfrac{a(x+1)+b}{(x+1)^2} = \dfrac{ax+(a+b)}{(x+1)^2}$ Equating coefficients: $a+b=3$ $a=5$ $b=-2$ Solving gives $a=5,\ \ b=-2$
Specific behaviours
✓ forms the equivalence of numerators correctly ✓ solves for a, b correctly

(b) Hence determine $\displaystyle\int \frac{5x+3}{(x+1)^2}\,dx$. (4 marks)

Solution
$\displaystyle\int \frac{5x+3}{(x+1)^2}\,dx = \int \frac{5}{x+1} - \frac{2}{(x+1)^2}\,dx$ $= 5\ln\lvert x+1\rvert + \dfrac{2}{x+1} + k$
Specific behaviours
✓ re-writes the integrand correctly in terms of the partial fractions ✓ anti-differentiates the $(x+1)^{-1}$ term correctly using the absolute value of a natural logarithm ✓ anti-differentiates the $2(x+1)^{-2}$ term correctly ✓ uses a constant of integration

Question 5 **(5 marks)**

The graph of function $f\left(x\right)=\dfrac{\left(x+a\right)\left(x-b\right)}{2\left(x-c\right)}$ is shown below. The constants a, b and c are positive.



(a) Determine the values for a, b and c. Justify your answer for c. (3 marks)

Solution
The x intercepts are $x=-1,\ x=5 \quad \therefore \quad a=1,\ b=5$ Vertical asymptote is $x=3 \quad \therefore \quad c=3$
Specific behaviours
✓ states the value for a and b correctly ✓ states the value for c correctly ✓ provides justification for the value of c

The inclined asymptote for the graph of $y=f\left(x\right)$ is shown.

(b) Determine the equation for the inclined asymptote. (2 marks)

Solution
$f\left(x\right) = \dfrac{\left(x+1\right)\left(x-5\right)}{2\left(x-3\right)} = \dfrac{x^2-4x-5}{2\left(x-3\right)} = \dfrac{\left(x-1\right)\left(x-3\right)}{2\left(x-3\right)} - \dfrac{8}{2\left(x-3\right)} \quad \ldots\ (1)$ $= \dfrac{x-1}{2} - \dfrac{4}{\left(x-3\right)}$ Hence as $\lvert x\rvert \to \infty \quad f\left(x\right) \to \dfrac{x-1}{2}$ i.e. $y=\dfrac{x-1}{2}$ is the inclined asymptote.
Specific behaviours
✓ re-writes function $f\left(x\right)$ in terms of partial fractions or equivalent to statement (1) ✓ states the equation for the inclined asymptote correctly

Question 6 **(6 marks)**

(a) Solve the system of equations: (3 marks)

$$x + y + z = 4$$
$$x - y - z = 2$$
$$2x - 3y + z = 11.$$

Solution
Consider $(1)+(2) \quad \therefore \quad 2x = 6 \quad \therefore \quad x = 3$ Using $x = 3 \quad (1): \quad y + z = 1$ $(3): \quad -3y + z = 5$ $(1)-3: \quad 4y = -4$ $\therefore \quad y = -1$ $\therefore \quad z = 2$ Hence the solution is $x = 3$ $y = -1$ $z = 2$
Specific behaviours
✓ eliminates a variable correctly using an appropriate technique ✓ solves correctly for the first variable ✓ solves correctly for the second and third variables

The third equation in part (a) on page 8 is changed to $2x - ky + z = 11$ where k is a real constant. The first two equations remain unchanged.

Ryan decided to solve this changed system of equations and obtained correctly the statement $(k+1)y = -4$.

(b) Determine the value of the constant k so that the changed system of equations does not have a unique solution. (1 mark)

Solution
$(k+1)y = -4$ Hence, for there to be no unique solution we require $k+1=0$ i.e. $k=-1$
Specific behaviours
✓ states that $k=-1$

(c) For the value of k determined from part (b), state the geometric interpretation of the solution of the three simultaneous equations. (2 marks)

Solution
For $k=-1$ there is NO solution to the simultaneous equations. Since the planes represented by the equations are NOT parallel, then the geometric significance is that the non-parallel planes have no intersection. i.e. the planes in pairs intersect in lines, but that these lines do not intersect.
Specific behaviours
✓ states that there is no intersection ✓ states that the planes are not parallel OR states the planes in pairs intersect in parallel lines

Question 7 **(5 marks)**

Consider the quartic polynomial $R(z) = z^4 - 6z^3 + 17z^2 - 22z + 14$ and $P(z) = z^2 - 2z + 2$ where $R(z) = P(z)(z^2 + az + b)$.

(a) Show that $(z - 1 - i)$ is a factor of $P(z)$. (2 marks)

Solution
$P(1+i) = (1+i)^2 - 2(1+i) + 2$ $= 2i - 2 - 2i + 2 = 0 \quad \therefore \; P(1+i) = 0$ Hence $(z-1-i)$ is a factor of $P(z)$.
Specific behaviours
✓ substitutes $z = 1 + i$ correctly into $P(z)$ ✓ expands correctly to show that $P(1+i) = 0$

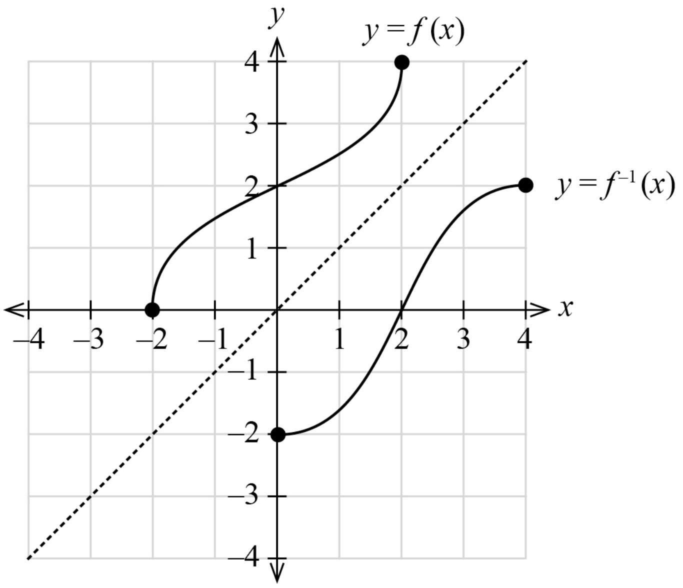
(b) Solve the equation $R(z) = 0$. (3 marks)

Solution
$z = 1 - i$ is also a root of $P(z)$ and $R(z)$. $R(z) = z^4 - 6z^3 + 17z^2 - 22z + 14 = (z^2 - 2z + 2)(z^2 + az + b)$ $\therefore \; 2b = 14$ i.e. $b = 7$ $-6z^3 = z^2(az) - 2z(z^2)$ i.e. $a = -4$ Solve $R(z) = (z^2 - 2z + 2)(z^2 - 4z + 7) = 0$ $\therefore \; z = 1 \pm i$ and $(z^2 - 4z + 7) = 0$ $(z-2)^2 + 3 = 0$ $\therefore \; z = 2 \pm \sqrt{3}i$
Specific behaviours
✓ states that $z = 1 + i$ and $z = 1 - i$ are solutions ✓ determines that $a = -4$ and $b = 7$ or $(z^2 - 4z + 7)$ is a factor of $R(z)$ ✓ solves the equation $z^2 + ax + b = 0$ correctly

Question 8 **(13 marks)**

The statement $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ can be written as $\frac{\pi}{3} = \cos^{-1}\left(\frac{1}{2}\right)$, where $\cos^{-1}$ represents the inverse cosine function.

The graph of $y = f(x) = \frac{4}{\pi}\cos^{-1}\left(-\frac{x}{2}\right)$ for $-2 \le x \le 2$ is shown below.



(a) Explain why $y = f^{-1}(x)$ exists. (1 mark)

Solution
Function $f(x)$ is one-to-one over its domain.
Specific behaviours
✓ states that function $f(x)$ is one-to-one

(b) Determine the defining rule for $y = f^{-1}(x)$. (2 marks)

Solution
$f: \ y = \frac{4}{\pi}\cos^{-1}\left(-\frac{x}{2}\right)$ $f^{-1}: \ x = \frac{4}{\pi}\cos^{-1}\left(-\frac{y}{2}\right)$ i.e. $\frac{\pi x}{4} = \cos^{-1}\left(-\frac{y}{2}\right)$ i.e. $-\frac{y}{2} = \cos\left(\frac{\pi x}{4}\right)$ $\therefore \ f^{-1}(x) = -2\cos\left(\frac{\pi x}{4}\right)$
Specific behaviours
✓ interchanges x, y to form the inverse relation ✓ determines the correct defining rule for $f^{-1}(x)$

Question 8 (continued)

(c) Sketch the graph of $y = f^{-1}(x)$ on the axes on page 12. (2 marks)

Solution
Shown on page 11
Specific behaviours
✓ indicates a curve that is a reflection of $y = f(x)$ about the line $y = x$ ✓ indicates the points $(0,-2),(2,0),(4,2)$

(d) If $y = \frac{4}{\pi}\cos^{-1}\left(-\frac{x}{2}\right)$, using implicit differentiation, show that $\frac{dy}{dx} = \frac{4}{\pi\sqrt{4-x^2}}$.

(5 marks)

Solution

$y = \frac{4}{\pi}\cos^{-1}\left(-\frac{x}{2}\right)$ i.e. $\frac{\pi y}{4} = \cos^{-1}\left(-\frac{x}{2}\right)$

i.e. $-\frac{x}{2} = \cos\left(\frac{\pi y}{4}\right)$

Differentiating implicitly with respect to x:

$$\frac{d}{dx}\left(-\frac{x}{2}\right) = \frac{d}{dx}\left(\cos\left(\frac{\pi y}{4}\right)\right)$$

$$-\frac{1}{2} = -\sin\left(\frac{\pi y}{4}\right)\times\frac{\pi}{4}\times\frac{dy}{dx}$$

Hence $\frac{dy}{dx} = \frac{1}{2}\times\frac{4}{\pi}\times\frac{1}{\sin\left(\frac{\pi y}{4}\right)} = \frac{2}{\pi\sin\left(\frac{\pi y}{4}\right)}$

We have that $\sin^2\left(\frac{\pi y}{4}\right)+\cos^2\left(\frac{\pi y}{4}\right)=1$

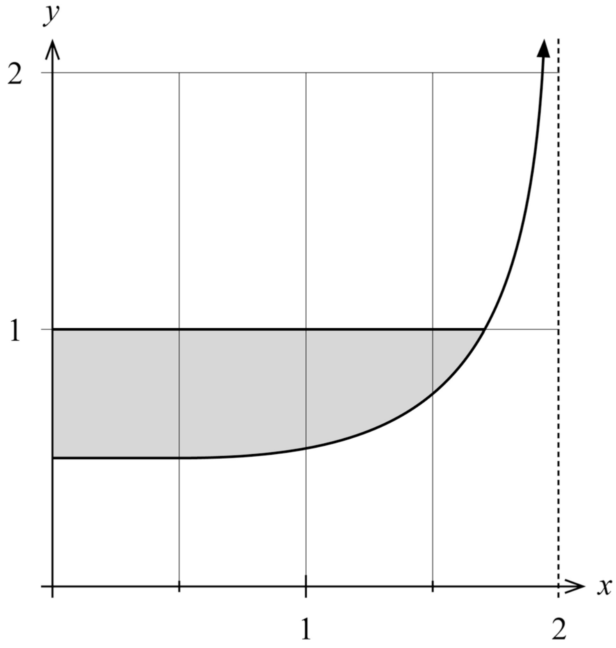
$$\therefore \sin\left(\frac{\pi y}{4}\right) = \sqrt{1-\cos^2\left(\frac{\pi y}{4}\right)} = \sqrt{1-\left(-\frac{x}{2}\right)^2}$$

$$\therefore \frac{dy}{dx} = \frac{2}{\pi\sin\left(\frac{\pi y}{4}\right)} = \frac{2}{\pi\sqrt{1-\frac{x^2}{4}}} = \frac{2}{\pi\sqrt{\frac{4-x^2}{4}}} = \frac{4}{\pi\sqrt{4-x^2}}$$

Specific behaviours

✓ writes an equivalent form of $y = f(x)$ in terms of $x = g(y)$ correctly
✓ differentiates the left-hand side correctly
✓ implicitly differentiates the right-hand side using the chain rule correctly
✓ determines the expression for $\frac{dy}{dx}$ in terms of y correctly
✓ uses the Pythagorean condition to express $\sin\left(\frac{\pi y}{4}\right)$ as $\sqrt{1-\left(-\frac{x}{2}\right)^2}$

The graph of $y = \frac{1}{\sqrt{4-x^2}}$ is shown below for $0 \le x < 2$.



The shaded region is bounded by the curve $y = \frac{1}{\sqrt{4-x^2}}$, the line $y = 1$ and the y axis.

(e) Determine the exact area of the shaded region. (3 marks)

Solution
From part (d) we have that $\frac{d}{dx}\left(\frac{4}{\pi}\cos^{-1}\left(-\frac{x}{2}\right)\right) = \frac{4}{\pi\sqrt{4-x^2}}$ Hence $\int \frac{1}{\sqrt{4-x^2}}dx = \cos^{-1}\left(-\frac{x}{2}\right) + k$ Line $y = 1$ intersects the curve when $1 = \frac{1}{\sqrt{4-x^2}}$ i.e. $x = \sqrt{3}$ $Area = \int_0^{\sqrt{3}}\left(1 - \frac{1}{\sqrt{4-x^2}}\right)dx = \left[x - \cos^{-1}\left(-\frac{x}{2}\right)\right]_0^{\sqrt{3}}$ $= \left(\sqrt{3} - \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) - \left(0 - \cos^{-1}(0)\right)$ $= \left(\sqrt{3} - \left(\frac{5\pi}{6}\right)\right) - \left(-\frac{\pi}{2}\right)$ $= \sqrt{3} - \frac{\pi}{3}$ sq. units
Specific behaviours
✓ forms the correct expression for the area as a definite integral ✓ determines the correct anti-derivative using part (d) ✓ obtains the exact area correctly

Published by the School Curriculum and Standards Authority of Western Australia
303 Sevenoaks Street
CANNINGTON WA 6107