2024/81899

Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS METHODS

Calculator-free

ATAR course examination 2024

Marking key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

Section One: Calculator-free **35% (51 Marks)**

Question 1 **(6 marks)**

(a) Differentiate the function $f(x) = x^2 \ln(4x+3)$. (2 marks)

Solution
$f'(x) = 2x\ln(4x+3) + \dfrac{4x^2}{4x+3}$
Specific behaviours
✓ applies the product rule ✓ obtains the correct expression

(b) Determine a fully simplified expression for $g(x)$, given that $g'(x) = \dfrac{3x}{3x^2+1}$ and $g(1) = \ln(6)$. (4 marks)

Solution
$g(x) = \int \dfrac{3x}{3x^2+1}\,dx$ $= \dfrac{1}{2}\int \dfrac{6x}{3x^2+1}\,dx$ $= \dfrac{1}{2}\ln\left(3x^2+1\right) + c$ Given that $g(1) = \ln(6)$ it follows that $\ln(6) = \dfrac{1}{2}\ln(4) + c$ $\Rightarrow c = \ln(6) - \ln\left(4^{\frac{1}{2}}\right)$ $= \ln(6) - \ln(2)$ $= \ln\left(\dfrac{6}{2}\right)$ $= \ln(3)$ Hence, $g(x) = \dfrac{1}{2}\ln\left(3x^2+1\right) + \ln(3)$ $= \ln\left(3\sqrt{3x^2+1}\right)$
Specific behaviours
✓ integrates correctly ✓ substitutes $g(1) = \ln(6)$ and correctly solves for c ✓ states expression for $g(x)$ ✓ applies log laws to fully simplify

Question 2 **(10 marks)**

(a) Determine the velocity of the graphic when it first appears on the screen. (2 marks)

Solution
The velocity of the graphic is given by $$v(t) = \frac{d}{dt}\left(\frac{1}{3}t^3 - 7t^2 + 40t\right)$$ $$= t^2 - 14t + 40$$ Hence, when it first appears on the screen $$v(0) = 40 \text{ cm/s}$$ The velocity of the graphic is 40 cm/s to the right of the screen.
Specific behaviours
✓ determines correct expression for $v(t)$ ✓ correctly evaluates $v(0)$ to obtain correct answer

(b) Is the graphic initially speeding up or slowing down? Justify your answer. (2 marks)

Solution
The acceleration of the graphic is given by $$a(t) = \frac{d}{dt}\left(t^2 - 14t + 40\right)$$ $$= 2t - 14$$ Hence, $a(0) = -14$ cm/s^2. The graphic is initially slowing down because the acceleration is in the opposite direction to the velocity.
Specific behaviours
✓ determines correct expression for $a(t)$ ✓ concludes that the graphic is slowing down with correct justification (only saying the acceleration is negative is not correct justification)

(c) Evaluate $\int_3^9 v(t)\,dt$ and explain what this integral represents. (3 marks)

Solution
$$\int_3^9 v(t)\,dt = x(9) - x(3)$$ $$= 36 - 66$$ $$= -30 \text{ cm}$$ The integral represents the change in displacement/position of the graphic from 3 seconds up to 9 seconds after the graphic appears on screen.
Specific behaviours
✓ correctly evaluates the integral ✓ states that the integral represents a change in displacement/position ✓ specifies that the change in displacement is from the 3 second to 9 second marks

Question 2 (continued)

(d) Calculate the total distance travelled by the graphic from the time it enters the screen to the time it leaves the screen 15 seconds later. (3 marks)

Solution
The graphic is at rest when $v(t) = 0$ $\Rightarrow 0 = t^2 - 14t + 40$ $= (t-4)(t-10)$ $\Rightarrow t = 4$ or $t = 10$ Hence, the distance d is $d = \lvert x(4) - x(0) \rvert + \lvert x(10) - x(4) \rvert + \lvert x(15) - x(10) \rvert$ $= \left\lvert 69\frac{1}{3} - 0 \right\rvert + \left\lvert 33\frac{1}{3} - 69\frac{1}{3} \right\rvert + \left\lvert 150 - 33\frac{1}{3} \right\rvert$ $= 69\frac{1}{3} + 36 + 116\frac{2}{3}$ $= 222$ cm
Specific behaviours
✓ solves $v(t) = 0$ to determine the times at which the graphic is at rest ✓ states a correct expression for the distance ✓ correctly calculates the total distance travelled

Question 3 **(6 marks)**

(a) Complete the missing probability entries in each of the tables above. (2 marks)

Solution

x	1	2	3	4	5
$\mathrm{P}(X=x)$	0.2	0.15	**0.25**	**0.35**	0.05

x	1	2	3	4	5
$\mathrm{P}(X\le x)$	0.2	**0.35**	0.6	0.95	**1**

Specific behaviours

✓ correctly completes two blank entries in the tables
✓ correctly completes the remaining two blank entries

(b) Evaluate $\mathrm{P}(2\le X\le 4)$. (2 marks)

Solution

$$\begin{aligned}\mathrm{P}(2\le X\le 4)&=\mathrm{P}(X\le 4)-\mathrm{P}(X\le 1)\\&=0.95-0.2\\&=0.75\end{aligned}$$

Or

$$\begin{aligned}\mathrm{P}(2\le X\le 4)&=\mathrm{P}(X=2)+\mathrm{P}(X=3)+\mathrm{P}(X=4)\\&=0.15+0.25+0.35\\&=0.75\end{aligned}$$

Specific behaviours

✓ writes a correct probability statement in terms of individual/cumulative probabilities
✓ calculates correct probability

(c) Evaluate $\mathrm{P}(X=1\,|\,X\le 3)$. (2 marks)

Solution

$$\begin{aligned}\mathrm{P}(X=1\,|\,X\le 3)&=\frac{\mathrm{P}(X=1)}{\mathrm{P}(X\le 3)}\\&=\frac{0.2}{0.6}\\&=\frac{1}{3}\end{aligned}$$

Specific behaviours

✓ writes a correct probability statement in terms of individual/cumulative probabilities
✓ calculates correct probability

Question 4 **(6 marks)**

(a) The uniformly distributed continuous random variable X has an expected value of 6 and a maximum value of 9. Determine the variance of X. (3 marks)

Solution
The expected value of a uniformly distributed continuous random variable X is midway between the maximum and minimum values, so the probability density function is $$f(x)=\begin{cases}\frac{1}{6}, & 3\le x\le 9\\ 0, & \text{otherwise}\end{cases}$$ The variance of X is given by $$\begin{aligned}Var(X)&=\int_3^9 (x-6)^2 f(x)\,dx\\ &=\frac{1}{6}\int_3^9 (x-6)^2\,dx\\ &=\frac{1}{6}\left[\frac{(x-6)^3}{3}\right]_3^9\\ &=\frac{1}{6}\left(9-(-9)\right)\\ &=3\end{aligned}$$
Specific behaviours
✓ determines the correct value and domain of the probability density function ✓ writes a correct integral expression for the variance ✓ correctly calculates the variance

(b) The binomially distributed discrete random variable W has a mean of $\frac{1}{2}$ and a variance of $\frac{5}{12}$. Evaluate $P(W=1)$. (3 marks)

Solution
From the question $np=\frac{1}{2}$ and $np(1-p)=\frac{5}{12}$. It follows that $$\frac{1}{2}(1-p)=\frac{5}{12}$$ $$\Rightarrow 1-p=\frac{5}{6}$$ $$\Rightarrow p=\frac{1}{6}$$ And $$n\frac{1}{6}=\frac{1}{2}$$ $$\Rightarrow n=3$$ Hence, $$P(W=1)=\binom{3}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2$$ $$=3\times\frac{1}{6}\times\frac{25}{36}$$ $$=\frac{25}{72}$$
Specific behaviours
✓ correctly states two equations relating n and p ✓ correctly solves for n and p ✓ correctly calculates the probability

Question 5 **(8 marks)**

(a) Express $\log_a(0.5)$ in terms of p. (2 marks)

Solution
$\log_a(0.5) = \log_a\left(2^{-1}\right) = -\log_a(2)$ or $\log_a(0.5) = \log_a\left(\frac{1}{2}\right) = \log_a(1) - \log_a(2)$
From the graph $p = \log_a(2)$, and so $\log_a(0.5) = -p$
Specific behaviours
✓ applies log laws to obtain $\log_a(0.5) = -\log_a(2)$ or $\log_a(0.5) = \log_a(1) - \log_a(2)$
✓ obtains correct expression

(b) Evaluate a^{5p}. (2 marks)

Solution
From the graph $p = \log_a(2)$, hence,
$a^{5p} = a^{5\log_a(2)} = a^{\log_a\left(2^5\right)} = 2^5 = 32$
Specific behaviours
✓ applies log laws to obtain $5p = \log_a\left(2^5\right)$
✓ uses inverse relationship between logarithms and exponentials to obtain the correct answer

Alternative Solution
From the graph $p = \log_a(2)$, hence,
$a^p = 2$ $\Rightarrow \left(a^p\right)^5 = 2^5$ $\Rightarrow a^{5p} = 32$
Specific behaviours
✓ rearranges $p = \log_a(2)$ to determine $a^p = 2$
✓ uses index laws to obtain correct answer

(c) Solve $\log_a(x-3)=3p$ for x. (2 marks)

Solution
From the graph $$\log_a(8)=3p$$ Hence, $$\log_a(x-3)=\log_a(8)$$ $$\Rightarrow x-3=8$$ $$\Rightarrow x=11$$ Or $$a^{3p}=x-3$$ $$\Rightarrow a^{\log_a(8)}=x-3$$ $$\Rightarrow 8=x-3$$ $$\Rightarrow x=11$$
Specific behaviours
✓ determines that $\log_a(8)=3p$ ✓ correctly solves for x

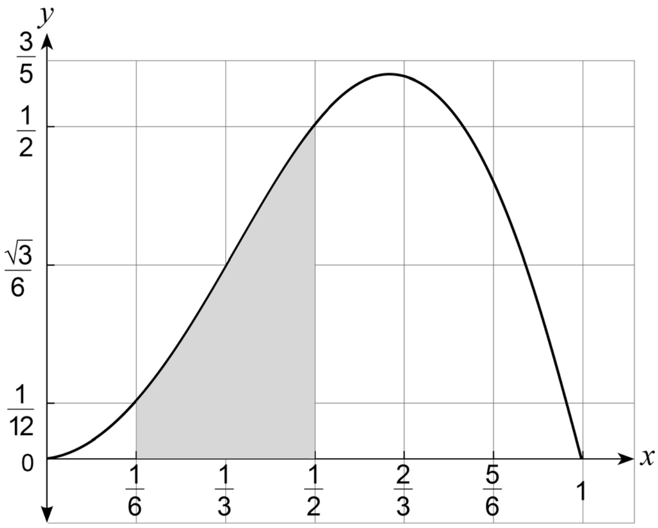
(d) Determine an equation for each of the **two** functions, A and B. (2 marks)

Solution
Function A is a vertical translation of $f(x)=\log_a(x)$ by an amount p upward. Hence, the equation for function A is $y=\log_a(x)+p$ or $y=\log_a(x)+\log_a(2)$ or $y=\log_a(2x)$ Function B is a horizontal translation of $f(x)=\log_a(x)$ by 1 unit to the left. Hence, the equation for function B is $$y=\log_a(x+1)$$
Specific behaviours
✓ determines a correct equation for function A ✓ determines the correct equation for function B

Question 6 **(5 marks)**

(a) On the diagram above, shade a region whose area is equal to $\int_{\frac{1}{6}}^{\frac{1}{2}} x\sin(\pi x)\, dx$. (1 mark)

Solution



Specific behaviours

✓ shades the correct region

(b) (i) By considering the areas of the rectangles shown in the graph of $y = x\sin(\pi x)$ above, demonstrate and explain why

$$\frac{1+2\sqrt{3}}{72} < \int_{\frac{1}{6}}^{\frac{1}{2}} x\sin(\pi x)\,dx < \frac{3+\sqrt{3}}{36}.$$

(3 marks)

Solution

Using the rectangles that underestimate the area (i.e. tops of rectangles lie below the graph)

$$\int_{\frac{1}{6}}^{\frac{1}{2}} x\sin(\pi x)\,dx > \frac{1}{6}\times\frac{1}{12}+\frac{1}{6}\times\frac{\sqrt{3}}{6}$$

$$=\frac{1}{6}\left(\frac{1+2\sqrt{3}}{12}\right)$$

$$=\frac{1+2\sqrt{3}}{72}$$

Using the rectangles that overestimate the area (i.e. tops of rectangles lie above the graph)

$$\int_{\frac{1}{6}}^{\frac{1}{2}} x\sin(\pi x)\,dx < \frac{1}{6}\times\frac{\sqrt{3}}{6}+\frac{1}{6}\times\frac{1}{2}$$

$$=\frac{1}{6}\left(\frac{3+\sqrt{3}}{6}\right)$$

$$=\frac{3+\sqrt{3}}{36}$$

Or

Using the rectangles that overestimate the area (i.e. sum of 4 rectangles shown on graph)

$$\int_{\frac{1}{6}}^{\frac{1}{2}} x\sin(\pi x)\,dx < \frac{1}{6}\times\frac{1}{12}+\frac{1}{6}\left(\frac{\sqrt{3}}{6}-\frac{1}{12}\right)+\frac{1}{6}\times\frac{\sqrt{3}}{6}+\frac{1}{6}\left(\frac{1}{2}-\frac{\sqrt{3}}{6}\right)$$

$$=\frac{1}{6}\times\left(\frac{1}{12}+\frac{\sqrt{3}}{6}-\frac{1}{12}+\frac{\sqrt{3}}{6}+\frac{1}{2}-\frac{\sqrt{3}}{6}\right)$$

$$=\frac{1}{6}\left(\frac{\sqrt{3}+3}{6}\right)$$

$$=\frac{3+\sqrt{3}}{36}$$

Hence, $\dfrac{1+2\sqrt{3}}{72} < \displaystyle\int_{\frac{1}{6}}^{\frac{1}{2}} x\sin(\pi x)\,dx < \dfrac{3+\sqrt{3}}{36}$.

Specific behaviours

✓ approximates the integral using an underestimate
✓ approximates the integral using an overestimate
✓ explains why the first approximation is an overestimate and the second is an underestimate

Question 6 (continued)

(ii) State **one** suggestion as to how the approximation from part (b)(i) could be improved. (1 mark)

Solution
The approximation could be improved by dividing the region between $x = \frac{1}{6}$ and $x = \frac{1}{2}$ into a larger number of narrower rectangles.
Specific behaviours
✓ suggests a correct approach

Question 7 **(10 marks)**

(a) Determine the speed of the bicycle at the end of the ramp, if the ramp angle is 45°. (2 marks)

Solution
A ramp angle of 45° corresponds to $\frac{\pi}{4}$ radians. Hence, $s\left(\frac{\pi}{4}\right)=\sqrt{\frac{101\sin\left(\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)}}$ $=\sqrt{\frac{101\left(\frac{1}{\sqrt{2}}\right)-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}}$ $=\sqrt{100}$ $=10$ m/s
Specific behaviours
✓ correctly evaluates $\sin\left(\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$ (or $\sin\left(45^\circ\right)=\cos\left(45^\circ\right)=\frac{1}{\sqrt{2}}$) ✓ calculates correct speed

(b) Determine $\frac{d}{d\theta}\left(\frac{101\sin(\theta)-\cos(\theta)}{\sin(\theta)}\right)$. Simplify your answer. (3 marks)

Solution
Using the quotient rule $\frac{d}{d\theta}\left(\frac{101\sin(\theta)-\cos(\theta)}{\sin(\theta)}\right)=\frac{\left(101\cos(\theta)+\sin(\theta)\right)\sin(\theta)-\cos(\theta)\left(101\sin(\theta)-\cos(\theta)\right)}{\sin^2(\theta)}$ $=\frac{101\cos(\theta)\sin(\theta)+\sin^2(\theta)-101\cos(\theta)\sin(\theta)+\cos^2(\theta)}{\sin^2(\theta)}$ $=\frac{\sin^2(\theta)+\cos^2(\theta)}{\sin^2(\theta)}$ $=\frac{1}{\sin^2(\theta)}$
Specific behaviours
✓ correctly differentiates using the quotient rule ✓ simplifies numerator to $\sin^2(\theta)+\cos^2(\theta)$ ✓ obtains correct simplified result

Question 7 (continued)

(c) Hence, show that $\frac{ds}{d\theta} = \frac{1}{2\sin^2(\theta)}\sqrt{\frac{\sin(\theta)}{101\sin(\theta) - \cos(\theta)}}$. (2 marks)

Solution
Using the result from part (b) and the chain rule gives $$\frac{ds}{d\theta} = \frac{1}{2}\left(\frac{101\sin(\theta) - \cos(\theta)}{\sin(\theta)}\right)^{-\frac{1}{2}} \frac{1}{\sin^2(\theta)} = \frac{1}{2\sin^2(\theta)}\sqrt{\frac{\sin(\theta)}{101\sin(\theta) - \cos(\theta)}}$$
Specific behaviours
✓ correctly uses result from part (b) with the chain rule to determine $\frac{ds}{d\theta}$ ✓ simplifies to obtain desired result

(d) Use the increments formula to estimate the change in s if the ramp angle is changed from 45° to 46°. (3 marks)

Solution
An increment of $1°$ corresponds to $\delta\theta = \frac{\pi}{180}$. Evaluating $\frac{ds}{d\theta}$ at $\theta = \frac{\pi}{4}$ gives $$\frac{ds}{d\theta}\left(\frac{\pi}{4}\right) = \frac{1}{2\sin^2\left(\frac{\pi}{4}\right)}\sqrt{\frac{\sin\left(\frac{\pi}{4}\right)}{101\sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right)}}$$ $$= \frac{1}{2\left(\frac{1}{\sqrt{2}}\right)^2}\sqrt{\frac{\frac{1}{\sqrt{2}}}{101\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}}\right)}}$$ $$= \sqrt{\frac{\left(\frac{1}{\sqrt{2}}\right)}{\left(\frac{100}{\sqrt{2}}\right)}}$$ $$= \frac{1}{\sqrt{100}}$$ $$= \frac{1}{10}$$ Hence, by the increments formula $$\delta s \approx \frac{ds}{d\theta}\left(\frac{\pi}{4}\right)\delta\theta$$ $$= \frac{1}{10} \times \frac{\pi}{180}$$ $$= \frac{\pi}{1800} \text{ m/s}$$
Specific behaviours
✓ correctly states the ramp angle increment of $\delta\theta = \frac{\pi}{180}$ ✓ correctly evaluates $\frac{ds}{d\theta}\left(\frac{\pi}{4}\right)$ ✓ implements increments formula to obtain correct value of δs

An *Acknowledgements variation* document is available on the Authority website.

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