Government of **Western Australia**
School Curriculum and Standards Authority

Sample Assessment Tasks

Mathematics Specialist
ATAR Year 12

Disclaimer

Any resources such as texts, websites and so on that may be referred to in this document are provided as examples of resources that teachers can use to support their learning programs. Their inclusion does not imply that they are mandatory or that they are the only resources relevant to the course.

School name

Mathematics Specialist
Unit 3

Test 1

Student name: ________________________ **Teacher name:** ________________________

Class: ________

Time allowed for this task:	55 minutes, in class, under test conditions	
	Section One – calculator-free section – 35 minutes	(30 marks)
	Section Two – calculator-assumed section – 20 minutes	(20 marks)
Materials required:	Calculator with CAS capability (to be provided by the student)	
Standard items:	Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlights	
Special items:	Drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in WACE examinations	
Marks available:	50 marks	
Task weighting:	**5%**	

Section One – calculator-free section **(30 marks)**

Question 1 (3.1.6, 3.1.15) (8 marks)

(a) Given $z = \sqrt{3} + i$ evaluate z^6 giving the answer in Cartesian form. (2 marks)

(b) (4 marks)

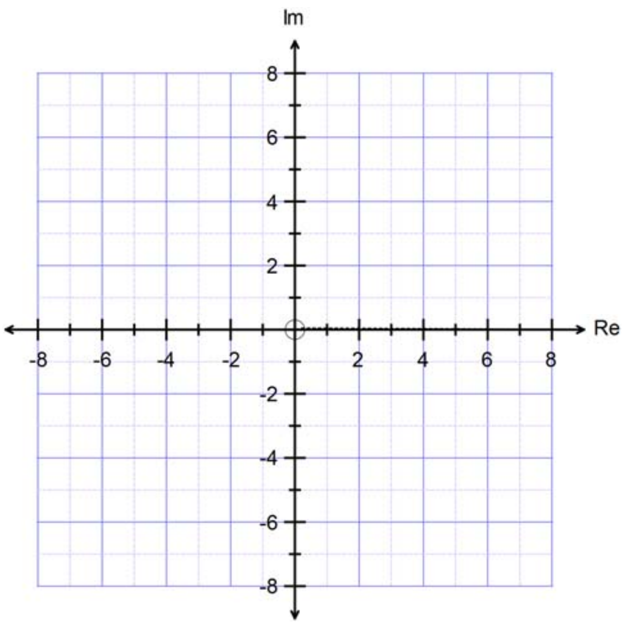
Given $Z_1 = cis\left(\frac{\pi}{3}\right)$ and $Z_2 = cis\left(\frac{\pi}{4}\right)$ evaluate the following in exact Cartesian form:

(i) $\overline{Z_1}$ (ii) iZ_2 (iii) $cis\left(\frac{\pi}{12}\right)$

(c) Solve $x^2 - 6x + 13 = 0$ for $x \in \mathrm{Im}$ in exact form. (2 marks)

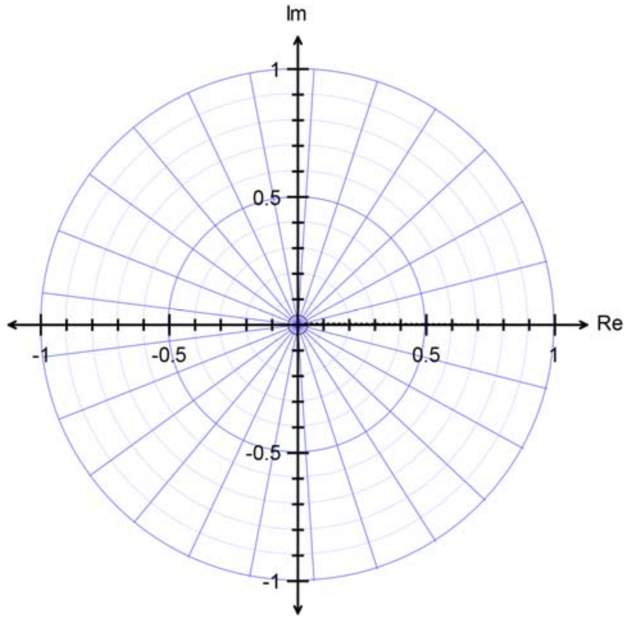
Question 2 (3.1.10) (6 marks)

(a) Sketch the set of points defined by $\left|z - (2 + 3i)\right| = \sqrt{13}$.



Question 3 (3.1.11, 3.1.12) (8 marks)

Determine and locate all solutions in the Argand plane to the equation $z^5 = 1$.



Question 4 (3.1.13, 3.1.15) (8 marks)

Given $H(z) = z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8$

(a) Evaluate $H(i)$, $H(-i)$ and $H(2)$. (3 marks)

(b) Hence, find all roots of the equation $z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8 = 0$. (5 marks)

Section Two – calculator-assumed section (20 marks)

Question 5 (3.1.7) **(10 marks)**

(a) Expand and simplify the expression $F(\theta)=(\cos\theta+i\sin\theta)^5$. (2 marks)

(b) Hence, express the $\text{Re}(F)$ in terms of $\cos\theta$. (3 marks)

(c) Use $\text{Re}(F)$ to solve the equation $16x^5 - 20x^3 + 5x - 1 = 0$ and express the solutions in trigonometric form. (5 marks)

Question 6 (3.1.7) **(10 marks)**

Given $z=\cos\theta+i\sin\theta$:

(a) Express $\dfrac{\left(z-\dfrac{1}{z}\right)}{i\left(z+\dfrac{1}{z}\right)}$ in trigonometric form. (4 marks)

(b) Show $z^2+\dfrac{1}{z^2}=2\cos 2\theta$ and hence prove $\cos 2\theta = 2\cos^2\theta - 1$. (6 marks)

Solutions and marking key for Test 1 for concurrent Unit 3 and Unit 4 program

Section One – calculator-free section **(30 marks)**

Question 1 (3.1.6, 3.1.15) **(8 marks)**

(a) Given $z=\sqrt{3}+i$ evaluate z^6 giving the answer in Cartesian form. (2 marks)

Given $z=\sqrt{3}+i$ evaluate z^6

$$z^6=\left(\sqrt{3}+i\right)^6=\sqrt{3}^6+6.\left(\sqrt{3}\right)^5.(i)^1+15.\left(\sqrt{3}\right)^4.(i)^2+20.\left(\sqrt{3}\right)^3.(i)^3+15.\left(\sqrt{3}\right)^2.(i)^4+6.\left(\sqrt{3}\right)^1.(i)^5+(i)^6$$

$$=27-135+45-1+\left(54\sqrt{3}-60\sqrt{3}+6\sqrt{3}\right)i$$

$$=-64$$

OR $z=2cis\left(\frac{\pi}{6}\right)\Rightarrow z^6=64cis(\pi)=-64$

Specific behaviours	Mark	Item
Expands the Cartesian form of z^6	1	simple
Simplifies correctly	1	simple
Or		
Expresses z^6 in polar form	1	simple
Expresses the answer in Cartesian form	1	simple

(b) (4 marks)

Given $Z_1=cis\left(\frac{\pi}{3}\right)$ and $Z_2=cis\left(\frac{\pi}{4}\right)$ evaluate the following in exact Cartesian form:

(i) $\overline{Z_1}$ (ii) iZ_2 (iii) $cis\left(\frac{\pi}{12}\right)$

Given $Z_1=cis\left(\frac{\pi}{3}\right)$ and $Z_2=cis\left(\frac{\pi}{4}\right)$ evaluate the following in exact Cartesian form:

(i) $\overline{Z_1}=\frac{1-\sqrt{3}i}{2}$ (ii) $iZ_2=\frac{-\sqrt{2}+\sqrt{2}i}{2}$

(iii) $cis\left(\frac{\pi}{12}\right)=\frac{Z_1}{Z_2}=\frac{1+\sqrt{3}i}{2}\times\frac{2}{\sqrt{2}+\sqrt{2}i}=\left(\frac{\left(\sqrt{2}+\sqrt{6}\right)+\left(\sqrt{6}-\sqrt{2}\right)i}{4}\right)$

Specific behaviours	Mark	Item
Writes the Cartesian form of $\overline{Z_1}$ correctly	1	simple
Writes the Cartesian form of iZ_2 correctly	1	simple
Expresses polar term for $\overline{Z_3}$ in Cartesian form	1	complex
Simplifies the Cartesian form correctly	1	complex

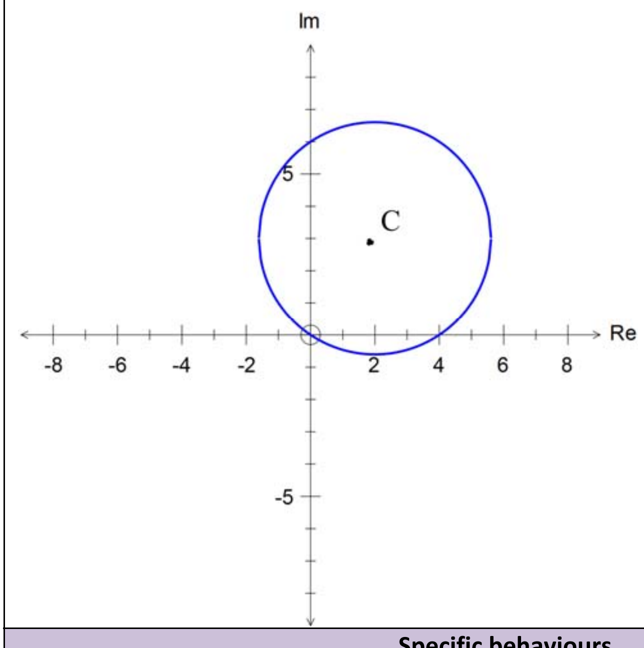
(c) Solve $x^2 - 6x + 13 = 0$ for $x \in \mathrm{Im}$ in exact form. (2 marks)

Solve $x^2 - 6x + 12 = 0$ $\Leftrightarrow \quad x^2 - 6x + 9 = -3$ $\Leftrightarrow \quad (x-3)^2 = -3$ $\Leftrightarrow \quad (x-3)^2 = 3i^2$ $\Leftrightarrow \quad x = 3 \pm \sqrt{3}\, i$	Or Solve $x^2 - 6x + 12 = 0$ $\Leftrightarrow \quad a = 1, b = -6$ and $c = 12$ $\Leftrightarrow \quad x = \dfrac{6 \pm \sqrt{36-48}}{2}$ $\Leftrightarrow \quad x = \dfrac{6 \pm \sqrt{-12}}{2} = 3 \pm \sqrt{3}i$

Specific behaviours	Mark	Item
Completes the square correctly	1	simple
Solves the equation using the exact form	1	simple
Or		
Uses the quadratic formula	1	simple
Simplifies the expressions to the correct exact form	1	simple

Question 2 (1.1.7) **(6 marks)**

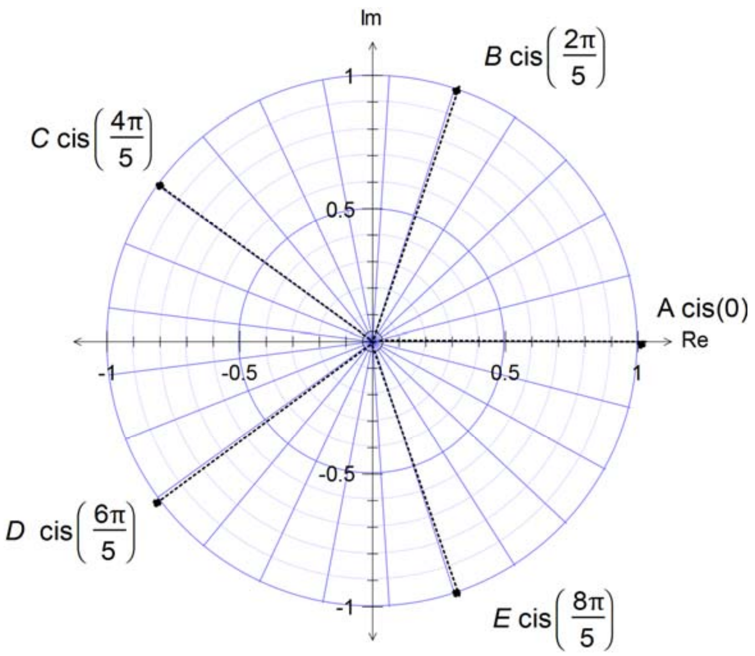
(a) Sketch the set of points defined by $\left|z - (2+3i)\right| = \sqrt{13}$.



Specific behaviours	Mark	Item
Draws a circle	1	simple
Has the correct centre *(2 + 3i)*	1	simple
Has the correct radius	1	simple
Circumference passes through *(0, 0) (4 + 0i)* and *(0 + 6i)*	3	simple

Question 3 (3.1.11, 3.1.12) (8 marks)

Determine and locate all solutions in the Argand plane to the equation $z^5 = 1$.



$$=1\times cis\left(0+\frac{2\pi\times 0}{5}\right)=1+0i$$
$$=1\times cis\left(0+\frac{2\pi\times 1}{5}\right)=cis\left(\frac{2\pi}{5}\right)$$
$$=1\times cis\left(0+\frac{2\pi\times 2}{5}\right)=cis\left(\frac{4\pi}{5}\right)$$
$$=1\times cis\left(0+\frac{2\pi\times 3}{5}\right)=cis\left(\frac{6\pi}{5}\right)$$
$$=1\times cis\left(0+\frac{2\pi\times 4}{5}\right)=cis\left(\frac{8\pi}{5}\right)$$

Specific behaviours	Mark	Item
Expresses the five solutions correctly	5	simple
Locates the solutions accurately on a polar graph	3	simple

Question 4 (3.1.13, 3.1.15) (8 marks)

Given $H(z)=z^5-2z^4+5z^3-10z^2+4z-8$:

(a) Evaluate $H(i)$, $H(-i)$ and $H(2)$ (3 marks)

Given $H(z)=z^5-2z^4+5z^3-10z^2+4z-8$

(a) Evaluate $H(i)$, $H(-i)$ and $H(2)$

$H(i)=i-2-5i+10+4i-8=0$

$H(-i)=-i-2+5i+10-4i-8=0$

$H(2)=32-32+40-40+8-8=0$

Specific behaviours	Mark	Item
Evaluates each of the three terms $H(i)$, $H(-i)$ and $H(2)$	3	simple

(b) Hence, find all roots of the equation $z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8 = 0$. (5 marks)

Given $H(z) = z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8$ from part (a)

$H(i) = 0$ $\Leftrightarrow (z-i)$ is a factor of $H(z)$

$H(-i) = 0$ $\Leftrightarrow (z+i)$ is a factor of $H(z)$

And $\Leftrightarrow (z^2+1)$ is a factor of $H(z)$

$H(2) = 0$ $\Leftrightarrow (z-2)$ is a factor of $H(z)$

Since $z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8 \div (z^2+1) = (z^3 - 2z^2 + 4z - 8)$

and $(z^3 - 2z^2 + 4z - 8) \div (z-2) = (z^2+4)$

then $z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8 = (z+i)(z-i)(z-2)(z+2i)(z-2i)$

Hence the roots to $z^5 - 2z^4 + 5z^3 - 10z^2 + 4z - 8 = 0$ are $z = \{\pm i, \pm 2i, 2\}$

Specific behaviours	Mark	Item
Uses the factor theorem to give factors $(z+i)(z-i)(z-2)$	1	simple
Determines the remaining factors $(z+2i)(z-2i)$	2	complex
Correctly writes all the roots	2	complex

Section Two – calculator-assumed section **(20 marks)**

Question 5 (3.1.7) **(10 marks)**

(a) Expand and simplify the expression $F(\theta)=(\cos\theta+i\sin\theta)^5$. (2 marks)

$$F(\theta) = (\cos\theta+i\sin\theta)^5$$
$$= (\cos^5\theta-10\cos^3\theta.\sin^2\theta+5\cos\theta.\sin\theta^4)+(\sin^5\theta+5\cos^4\theta.\sin\theta-10\cos^2\theta.\sin^3\theta)i$$

Specific behaviours	Mark	Item
Shows the real and imaginary terms correctly	2	simple

(b) Hence, express the $\text{Re}(F)$ in terms of $\cos\theta$. (3 marks)

$$\text{Re}(F) = \cos^5\theta-10\cos^3\theta.\sin^2\theta+5\cos\theta.\sin\theta^4$$
$$= \cos^5\theta-10\cos^3\theta(1-\cos^2\theta)+5\cos\theta(1-\cos^2\theta)^2$$
$$= \cos^5\theta-10\cos^3\theta+10\cos^5\theta+5\cos\theta(1-2\cos^2\theta+\cos^4\theta)$$
$$= 16\cos^5\theta-20\cos^3\theta+5\cos\theta$$

Specific behaviours	Mark	Item
Writes the real part of $F(\theta)$	1	simple
Substitutes for $\sin^2\theta=1-\cos^2\theta$	1	simple
Gives the correct expression for $\text{Re}(F)$	1	simple

(c) Use $\text{Re}(F)$ to solve the equation $16x^5 - 20x^3 + 5x - 1 = 0$ and express the solutions in trigonometric form. (5 marks)

$F(\theta) = (\cos\theta + i\sin\theta)^5$ $= \cos 5\theta + i\sin 5\theta$ {DeMoivre} $\Leftrightarrow \text{Re}(F) = \cos 5\theta$ $\text{Re}(F) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ from part (b) Hence $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$..........(1) To solve $16x^5 - 20x^3 + 5x - 1 = 0$ $\Leftrightarrow$ $16x^5 - 20x^3 + 5x = 1$ {Let $x = \cos\theta$} $\Leftrightarrow$ $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 1$ $\Leftrightarrow$ $\cos 5\theta = 1$ $\Leftrightarrow$ $5\theta = 0, 2\pi, \ldots$ $\Leftrightarrow$ $\theta = \frac{2\pi n}{5}, n = 0,1,2,3,4$ $\Leftrightarrow$ $x = \cos\left(\frac{2\pi n}{5}\right), n = 0,1,2,3,4$

Specific behaviours	Mark	Item
Uses De Moivre to state $\text{Re}(F) = \cos 5\theta$	1	complex
Makes the substitution $x = \cos\theta$ in polynomial	1	complex
Replaces the polynomial in $\cos\theta$ with $\cos 5\theta$	1	complex
Solves $\cos 5\theta = 1$ in terms of θ	1	complex
Gives all five solutions in terms of x	1	complex

Question 6 (3.1.7) **(10 marks)**

Given $z = \cos\theta + i\sin\theta$:

(a) Express $\dfrac{\left(z - \frac{1}{z}\right)}{i\left(z + \frac{1}{z}\right)}$ in trigonometric form. (4 marks)

$\dfrac{\left(z - \frac{1}{z}\right)}{i\left(z + \frac{1}{z}\right)} = \dfrac{(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)}{i\left((\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)\right)}$ $= \dfrac{2i\sin\theta}{2i\cos\theta}$ $= \tan\theta$

Specific behaviours	Mark	Item
Rewrites the complex numbers z and $\frac{1}{z}$ in trig form	2	simple
Simplifies both numerator and denominator Writes the correct final term	2	simple

(b) Show $z^2+\frac{1}{z^2}=2\cos 2\theta$ and hence prove $\cos 2\theta=2\cos^2\theta-1$ (6 marks)

$$z^2+\frac{1}{z^2}=(\cos 2\theta+i\sin 2\theta)+(\cos 2\theta-i\sin 2\theta)$$
$$=2\cos 2\theta\text{.....................................}(1)$$
$$z^2+\frac{1}{z^2}=(\cos\theta+i\sin\theta)^2+(\cos\theta-i\sin\theta)^2$$
$$=\left(\cos^2\theta+2i\sin\theta\cos\theta-\sin^2\theta\right)+\left(\cos^2\theta-2i\sin\theta\cos\theta-\sin^2\theta\right)$$
$$=2\left(\cos^2\theta-\sin^2\theta\right)\text{...................}(2)$$
$$(1)=(2)\Leftrightarrow 2\cos 2\theta=2\left(\cos^2\theta-\sin^2\theta\right)$$
$$\Leftrightarrow \cos 2\theta=2\cos^2\theta-1$$

Specific behaviours	Mark	Item
Rewrites z^2 and $\frac{1}{z^2}$ using double angle form 2θ	1	complex
Gathers terms and simplifies	1	complex
Rewrites z^2 and $\frac{1}{z^2}$ using single angle form θ	1	complex
Gathers terms and simplifies	1	complex
Equates both equations	1	complex
Writes correct final expression	1	complex

Question	1	2	3	4	5	6	Total
Simple	8	6	8	4	5	4	35
Complex	0	0	0	4	5	6	15
	8	6	8	8	10	10	50