Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS SPECIALIST

Calculator-free

ATAR course examination 2019

Marking key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2019/63011

Section One: Calculator-free **35% (47 Marks)**

Question 1 **(4 marks)**

Using the identity $2\sin A\cos B = \sin(A+B)+\sin(A-B)$, evaluate exactly the definite integral

$$\int_0^{\frac{\pi}{2}} 6\sin\left(\frac{5x}{2}\right)\cos\left(\frac{x}{2}\right)dx.$$

Solution
Using the given identity $2\sin A\cos B = \sin(A+B)+\sin(A-B)$ $6\sin\left(\frac{5x}{2}\right)\cos\left(\frac{x}{2}\right) = 6\times\frac{1}{2}\left(\sin\left(\frac{5x}{2}+\frac{x}{2}\right)+\sin\left(\frac{5x}{2}-\frac{x}{2}\right)\right)$ $= 3(\sin 3x+\sin 2x)$ $\int_0^{\frac{\pi}{2}} 6\sin\left(\frac{5x}{2}\right)\cos\left(\frac{x}{2}\right)dx = \int_0^{\frac{\pi}{2}}(3\sin 3x+3\sin 2x)\,dx$ $= \left[-\cos 3x-\frac{3\cos 2x}{2}\right]_0^{\frac{\pi}{2}}$ $= \left[-\cos\frac{3\pi}{2}-\frac{3\cos\pi}{2}\right]-\left[-\cos 0-\frac{3\cos 0}{2}\right]$ $= \left[-(0)-\frac{3(-1)}{2}\right]-\left[-1-\frac{3(1)}{2}\right]$ $= \left(\frac{3}{2}\right)-\left(-\frac{5}{2}\right) = 4$
Specific behaviours
✓ determines the factor 3 in relating the expressions   ✓ obtains the integrand terms $\sin 3x+\sin 2x$ ✓ anti-differentiates term by term correctly   ✓ evaluates the definite integral correctly

Question 2 **(6 marks)**

Consider the function $P(z) = z^4 - 2z^3 + 14z^2 - 8z + 40$, defined over the complex numbers.

(a) Show that $(z - 2i)$ is a factor of $P(z)$. (2 marks)

Solution
$P(2i) = (2i)^4 - 2(2i)^3 + 14(2i)^2 - 8(2i) + 40$ $= 16(1) - 16(-1)(i) + 14(4)(-1) - 16i + 40$ $= 16 + 16i - 56 - 16i + 40 \quad ... \ (1)$ $= 0$ Hence $(z - 2i)$ is a factor of $P(z)$.
Specific behaviours
✓ substitutes $z = 2i$ correctly into $P(z)$ ✓ obtains the 5 terms in expression (1) to deduce $P(2i) = 0$

(b) Hence or otherwise, solve the equation $P(z) = 0$, giving solutions in the form $a + bi$.
(4 marks)

Solution
Since $(z - 2i)$ is a factor then so is $(z + 2i)$. Hence $(z + 2i)(z - 2i) = (z^2 + 4)$ is also a factor of $P(z)$. $\therefore \ P(z) = (z^2 + 4)Q(z)$ where $Q(z) = z^2 - 2z + 10$ i.e. Solving $Q(z) = 0$ $\quad z^2 - 2z + 10 = 0$ $\quad$ OR $\quad \therefore \ (z^2 + 4) = 0$ $\therefore \ (z-1)^2 + 9 = 0$ $\quad \therefore \ z = \pm 2i$ $\therefore \ (z-1)^2 = -9$ $i.e. \quad z = 1 \pm 3i$
Specific behaviours
✓ deduces $(z + 2i)$ is a factor of $P(z)$ or states $z = -2i$ is a solution ✓ deduces $(z^2 + 4)$ is a factor of $P(z)$ ✓ factorises $P(z)$ as $(z^2 + 4)(z^2 - 2z + 10)$ ✓ states $z = 1 \pm 3i$ as solutions to $P(z) = 0$

Question 3 **(5 marks)**

(a) Given that $\dfrac{2x^2+5x+6}{x^2(x+3)} = \dfrac{a}{x} + \dfrac{b}{x^2} + \dfrac{c}{x+3}$, determine the values of a, b and c.

(2 marks)

Solution
$\dfrac{a}{x} + \dfrac{b}{x^2} + \dfrac{c}{x+3} = \dfrac{ax(x+3) + b(x+3) + cx^2}{x^2(x+3)}$ $= \dfrac{(a+c)x^2 + (3a+b)x + 3b}{x^2(x+3)}$ Hence equating co-efficients we obtain $a+c=2$ $3a+b=5$ $3b=6$ Solving obtains $a=1,\ \ b=2,\ c=1$
Specific behaviours
✓ obtains the numerator correctly in terms of a, b, c in simplifying the fractions   ✓ determines the values for a, b, c correctly

(b) Hence determine $\displaystyle\int \frac{2x^2+5x+6}{x^2(x+3)}\,dx$. (3 marks)

Solution
$\displaystyle\int \frac{2x^2+5x+6}{x^2(x+3)}dx = \int \frac{1}{x} + \frac{2}{x^2} + \frac{1}{x+3}dx$ $= \ln\lvert x\rvert - \dfrac{2}{x} + \ln\lvert x+3\rvert + c$
Specific behaviours
✓ expresses the integrand in terms of the partial fractions correctly   ✓ anti-differentiates correctly (using absolute value of the natural logarithm)   ✓ uses an integration constant

Question 4 **(7 marks)**

Functions f, g and h are defined such that:

$$f(x) = \frac{1}{x-1},\; g(x) = x^2,\; h(x) = \sqrt{x}.$$

(a) Determine the defining rule for $f(h(x))$. (1 mark)

Solution
$f(h(x)) = \dfrac{1}{\sqrt{x}-1}$
Specific behaviours
✓ states the correct defining rule

(b) Determine the domain for $f(h(x))$. (2 marks)

Solution
$D_{foh} = \{ x \mid x \geq 0,\; x \neq 1\}$
Specific behaviours
✓ states $x \geq 0$ ✓ states $x \neq 1$

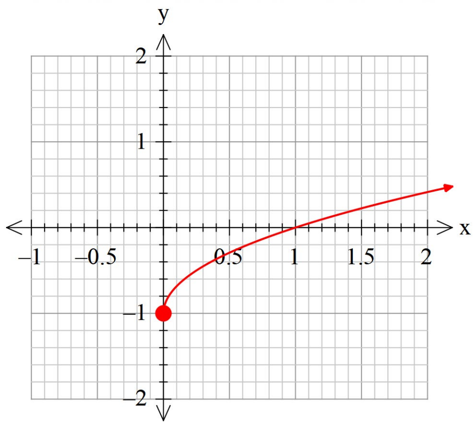
(c) Determine the range for $f(h(x))$. (2 marks)

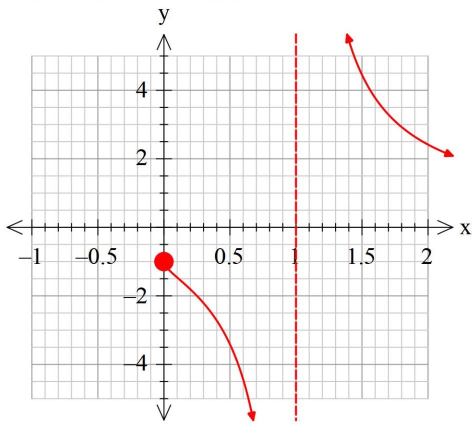
Solution
When $x > 1$ $\quad f(h(x)) > 0$ When $0 \leq x < 1$ $\quad -1 \leq \sqrt{x} - 1 < 0 \quad \therefore \quad -1 \geq \dfrac{1}{\sqrt{x}-1}$ Hence $R_{foh} = \{ y \mid y > 0 \;\cup\; y \leq -1\}$
Specific behaviours
✓ states $y > 0$ ✓ states $y \leq -1$

Question 4 (continued)

Alternative Solution

Graph $y = \sqrt{x} - 1$ and then graph its reciprocal function $y = f\left(h\left(x\right)\right) = \dfrac{1}{\sqrt{x} - 1}$





Hence $R_{foh} = \{\, y \mid y > 0 \ \cup \ y \le -1 \}$

Specific behaviours

✓ states $y > 0$

✓ states $y \le -1$

(d) Is it true that $f\left(h\left(g\left(x\right)\right)\right) = \dfrac{1}{x-1} = f\left(x\right)$? Justify your answer. (2 marks)

Solution

The statement is FALSE.

$h\left(g\left(x\right)\right) = \sqrt{x^2} = |x| \ \geq \ 0$

Hence $f\left(h\left(g\left(x\right)\right)\right) = \dfrac{1}{\sqrt{x^2} - 1} = \dfrac{1}{|x| - 1}$ $\quad D_{fohog} = \{\, x \mid x \in \mathbb{R},\ x \neq \pm 1 \}$

$R_{fohog} = \{\, y \mid y > 0 \ \cup \ y \le -1 \}$

But $f\left(x\right) = \dfrac{1}{x-1}$ $\quad D_{fohog} = \{\, x \mid x \in \mathbb{R},\ x \neq 1 \}$

$R_{fohog} = \{\, y \mid y > 0 \}$

$\therefore \quad f\left(h\left(g\left(x\right)\right)\right) \neq f\left(x\right)$ as they have different DOMAIN and RANGE values.

Specific behaviours

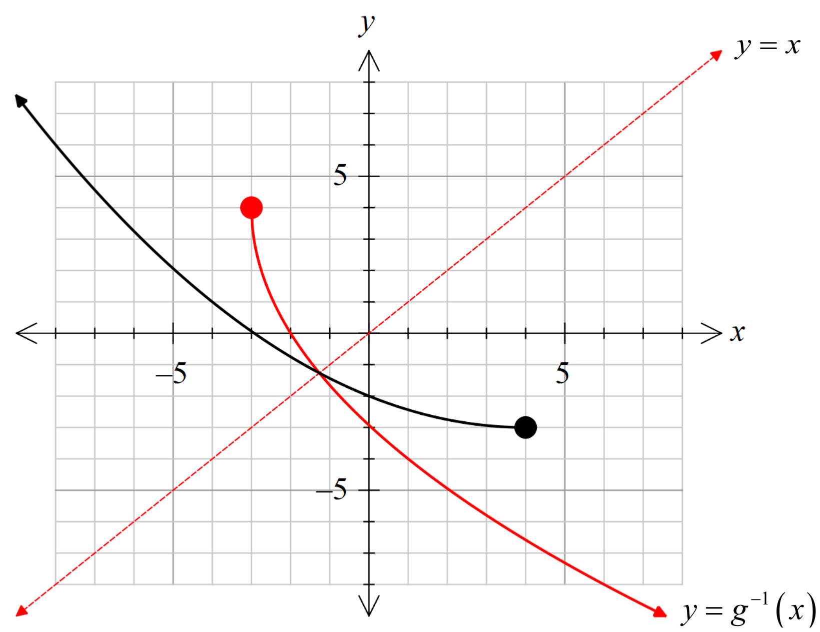
✓ states that the statement is false

✓ justifies the statement is false

Alternative Solution
The statement is FALSE. This would be true if $h\left(g\left(x\right)\right)=x$ i.e. true if $\sqrt{x^2}=x$. But actually $\sqrt{x^2} = \lvert x\rvert \neq x$.
Specific behaviours
✓ states that the statement is false ✓ justifies the statement is false

Question 5 **(6 marks)**

The graph of $y = g(x)$ is shown below.



(a) Sketch the graph of $y = g^{-1}(x)$ on the axes above. (3 marks)

Solution
See above graph axes.
Specific behaviours
✓ indicates the points $(-3,4)$, $(-2,0)$ and $(2,-5)$
✓ indicates the range $y \le 4$ (arrow on graph not required)
✓ indicates symmetry of $y = g^{-1}(x)$ with $y = g(x)$ about the line $y = x$

(b) Given that $g(x) = \frac{1}{16}(x-4)^2 - 3$ where $x \le 4$, determine the defining rule for $y = g^{-1}(x)$. (3 marks)

Solution
$g: \quad y = \frac{1}{16}(x-4)^2 - 3 \qquad g^{-1}: \quad x = \frac{1}{16}(y-4)^2 - 3$ $\therefore \quad 16(x+3) = (y-4)^2 \quad \ldots (1)$ $y - 4 = \pm\, 4\sqrt{x+3}$ Since $R_{g^{-1}} = D_g \ (x \le 4)$ then $g^{-1}(x) = 4 - 4\sqrt{x+3}$
Specific behaviours
✓ interchanges the x, y coordinates to obtain the inverse ✓ manipulates the equation correctly to obtain statement 1 ✓ writes the correct defining rule

Question 6 **(6 marks)**

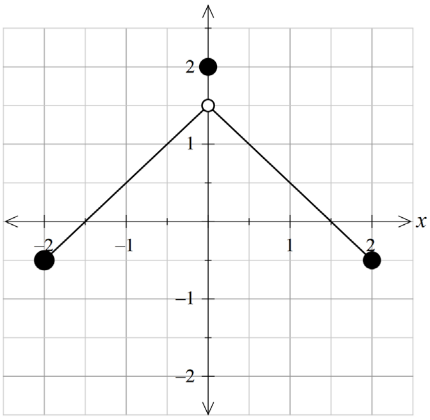
Using the substitution $x = 2\sin\theta$, evaluate exactly $\int_0^{\sqrt{3}} \sqrt{1-\frac{x^2}{4}}\,dx$.

Solution
When $x=0,\ \theta=0$ $\quad \frac{dx}{d\theta}=2\cos\theta \quad \therefore\ dx = 2\cos\theta d\theta$ and $x=\sqrt{3},\ \theta=\frac{\pi}{3}$ $\therefore \int_0^{\sqrt{3}} \sqrt{1-\frac{x^2}{4}}\,dx = \int_0^{\frac{\pi}{3}} \sqrt{1-\frac{4\sin^2\theta}{4}}\,(2\cos\theta\,d\theta)$ $= \int_0^{\frac{\pi}{3}} \sqrt{1-\sin^2\theta}\,(2\cos\theta\,d\theta)$ $= \int_0^{\frac{\pi}{3}} \sqrt{\cos^2\theta}\,(2\cos\theta\,d\theta)$ $= \int_0^{\frac{\pi}{3}} 2\cos^2\theta\,d\theta = \int_0^{\frac{\pi}{3}} (1+\cos 2\theta)\,d\theta$ $= \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{3}}$ $= \frac{\pi}{3} + \frac{\sqrt{3}}{4}$
Specific behaviours
✓ changes the limits correctly ✓ obtains dx in terms of $d\theta$ correctly ✓ simplifies the integrand correctly using the identity $\sin^2\theta + \cos^2\theta = 1$ ✓ uses the $\cos 2\theta$ identity to correctly re-write the integrand ✓ anti-differentiates the integrand correctly ✓ evaluates the definite integral correctly

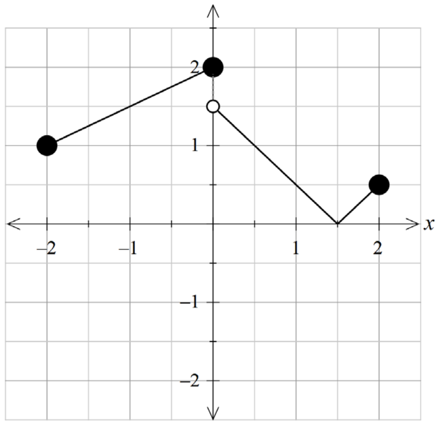
Question 7 **(4 marks)**

The graphs of $y = f(|x|)$ and $y = |f(x)|$ are shown below.

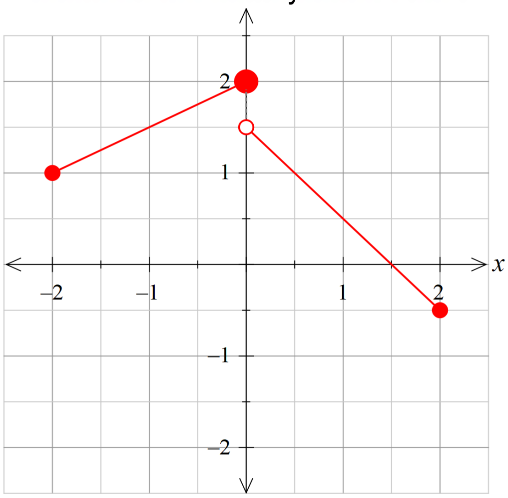
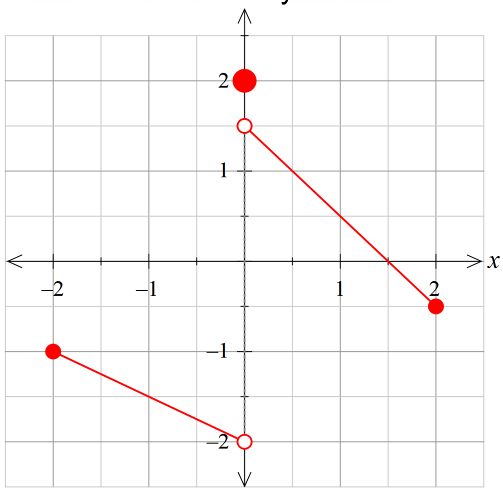
$f(|x|)$



$|f(x)|$



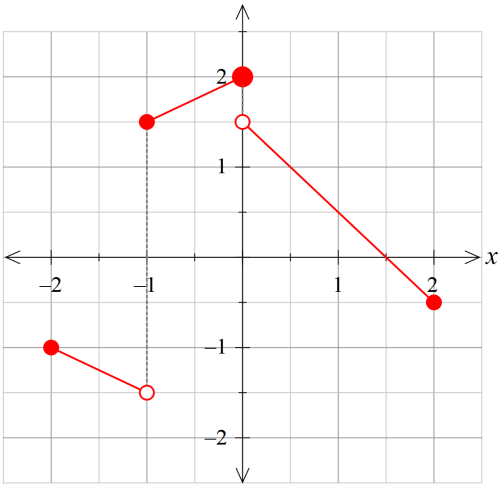
Given that $y = f^{-1}(x)$ is also a function, sketch a possible graph for $y = f(x)$ on the axes below. Justify your answer considering $y = f^{-1}(x)$.

Solution
There are many possibilities for $y = f(x)$. Two of these are:
Possibility A  Possibility B 
Since $y = f^{-1}(x)$ is a function then $y = f(x)$ over its domain $-2 \le x \le 2$ must be a ONE-TO-ONE function (which does not occur with possibility A or D). Hence $y = f(x)$ could be possibility B or C. Alternatively, function $f(x)$ must satisfy the 'horizontal' line test.
Specific behaviours
✓ indicates the points $(0,2)$, $(1.5,0)$, $(2,-0.5)$ ✓ indicates $y = 1.5 - x$ for $0 < x \le 2$ ✓ indicates $y = 0.5x + 2$ OR $y = -0.5x - 2$ for $-2 \le x < 0$ or equivalent to obtain $y = f(\|x\|)$ and $y = \|f(x)\|$ correctly ✓ justifies that $y = f(x)$ must be a one-to-one function so that $y = f^{-1}(x)$ is a function

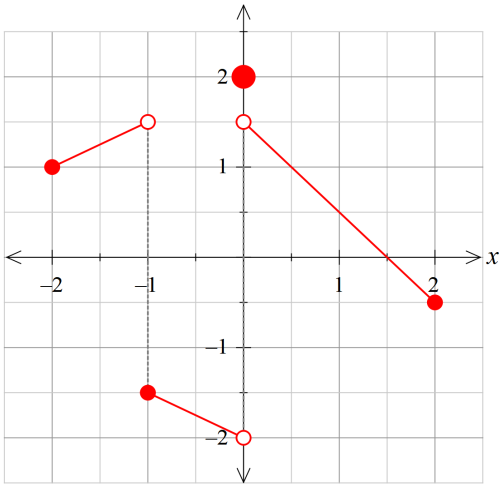
Alternative Solution

Other possibilities for $y = f(x)$:

Possibility C

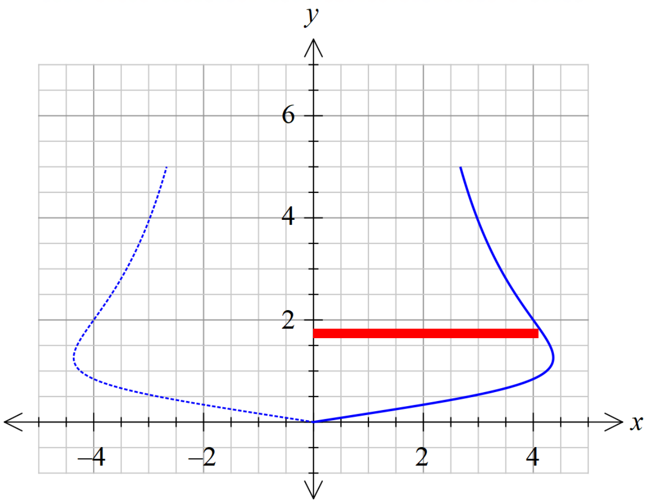


Possibility D



Question 8 **(5 marks)**

The top part of a wine glass is modelled by rotating the graph of $x^2 = y^2\left(36 - x^2 y\right)$ from $y = 0$ to $y = 5$ about the y axis as shown below. Dimensions are measured in centimetres.



(a) Show that the volume, V cm^3, when the glass is full is given by $V = \pi\int_0^5 \frac{36y^2}{1+y^3}\,dy$.

(1 mark)

Solution
From $x^2 = y^2\left(36 - x^2 y\right)$ $\therefore\ x^2 = 36y^2 - x^2y^3$ $\therefore\ x^2 + x^2y^3 = 36y^2$ i.e. $x^2\left(1+y^3\right) = 36y^2$ gives $x^2 = \frac{36y^2}{1+y^3} = \left(\frac{6y}{\sqrt{1+y^3}}\right)^2$ Hence $dV = \pi r^2 dy = \pi\left(\frac{6y}{\sqrt{1+y^3}}\right)^2 dy$. To obtain the TOTAL of all the possible thin cylindrical disks we add (integrate) over the interval of the possible y values i.e. integrate from $y = 0$ to $y = 5$. Hence volume $V = \int_0^5 dV = \int_0^5 \pi\left(\frac{6y}{\sqrt{1+y^3}}\right)^2 dy = \pi\int_0^5 \frac{36y^2}{1+y^3}\,dy$
Specific behaviours
✓ obtains the x coordinate correctly from the given curve equation (or x^2)

(b) Determine the exact volume V cm³. (4 marks)

Solution
$V = \int_0^5 \pi\left(\frac{6y}{\sqrt{1+y^3}}\right)^2 dy = \pi\int_0^5 \frac{36y^2}{1+y^3}dy$
Using $u = 1+y^3$ $\quad \frac{du}{dy} = 3y^2 \quad \therefore\ dy = \frac{du}{3y^2}$
When $y=0,\ u=1$ $\quad \therefore\ V = \pi\int_1^{126} 36y^2.\frac{1}{u}.\frac{du}{3y^2}$
$y=5,\ u=126$ $\quad = \pi\int_1^{126}\frac{12}{u}du = \pi\left[12\ln\lvert u\rvert\right]_1^{126}$
$= 12\pi(\ln 126)$
Specific behaviours
✓ changes the limits correctly ✓ obtains the integrand correctly in terms of u ✓ anti-differentiates correctly (absolute value of natural logarithm not required) ✓ obtains the exact value for the volume in terms of π and a natural logarithm correctly

Alternative Solution
$V = \int_0^5 \pi\left(\frac{6y}{\sqrt{1+y^3}}\right)^2 dy = \pi\int_0^5 \frac{36y^2}{1+y^3}dy$
$\therefore\ V = \pi\int_0^5 \frac{12(3y^2)}{1+y^3}dy$
$= 12\pi\int_0^5 \frac{d}{dy}(1+y^3).\frac{1}{1+y^3}dy$
$= 12\pi\left[\ln(1+y^3)\right]_0^5$
$= 12\pi(\ln 126)$
Specific behaviours
✓ recognises $36y^2$ as a multiple of the derivative of $1+y^3$ ✓ obtains the numerator as 12 times the derivative of $1+y^3$ ✓ anti-differentiates correctly (absolute value of natural logarithm not required) ✓ obtains the exact value for the volume in terms of π and a natural logarithm correctly

Question 9 **(4 marks)**

Consider the complex equation $z^n - 1 = 0$, where n is any positive integer $n \geq 3$.

If the roots are designated as $z_0,\ z_1,\ z_2,\ \ldots,\ z_{n-1}$, then determine the exact value for the product of the roots $p = z_0 \times z_1 \times z_2 \times\ \ldots\ \times z_{n-1}$.

Solution

$$z^n = 1 = cis(0) \quad \therefore \quad z = cis\left(\frac{0 + 2\pi k}{n}\right) = cis\left(\frac{2k\pi}{n}\right) \text{ where } k = 0, 1, 2, \ \ldots\ , n-1$$

$$\therefore \quad z_0 = cis(0) = 1,\ z_1 = cis\left(\frac{2\pi}{n}\right),\ z_2 = cis\left(\frac{4\pi}{n}\right),\ z_3 = cis\left(\frac{6\pi}{n}\right),\ z_4 = cis\left(\frac{8\pi}{n}\right)$$

$$z_{n-1} = cis\left(\frac{2(n-1)\pi}{n}\right)$$

$$p = cis(0)\,cis\left(\frac{2\pi}{n}\right)cis\left(\frac{4\pi}{n}\right)cis\left(\frac{6\pi}{n}\right)\ \ldots\ cis\left(\frac{2(n-1)\pi}{n}\right)$$

$$= cis\left(0 + \frac{2\pi}{n} + \frac{4\pi}{n} + \frac{6\pi}{n} + \ldots \frac{2(n-1)\pi}{n}\right)$$

$$= cis\left(\frac{2\pi}{n}\left(1 + 2 + 3 + \ldots(n-1)\right)\right)$$

$$= cis\left(\frac{2\pi}{n} \times \frac{(n-1)(n)}{2}\right)$$

$$= cis\left((n-1)\pi\right) = \cos(n-1)\pi + i\sin(n-1)\pi$$

Since $\sin(n-1)\pi = 0$ for all integer values of n and $\cos(n-1)\pi = \pm 1$, then

Product $p = 1$ if n is ODD

$p = -1$ if n is EVEN.

Specific behaviours

✓ expresses the roots in the form $cis\left(\frac{2k\pi}{n}\right)$ where $k = 0, 1, 2,\ \ldots\ , n-1$

✓ forms the product $p = cis\left(\frac{2\pi}{n}\right)cis\left(\frac{4\pi}{n}\right)\ldots cis\left(\frac{2(n-1)\pi}{n}\right)$ correctly

✓ uses DeMoivres Theorem to obtain $cis\left((n-1)\pi\right)$ correctly

✓ states the two possible values correctly for n even and odd

Alternative Solution
Equation is $z^n - 1 = 0$ Given that the roots are: $z_0,\ z_1,\ z_2,\ \ldots,\ z_{n-1}$ means that the equation can be written in the form $(z-z_0)(z-z_1)(z-z_2)\ldots(z-z_{n-1}) = 0$ i.e. $(z-z_0)(z-z_1)(z-z_2)\ldots(z-z_{n-1}) = z^n - 1$ Hence the LHS constants $(-z_0)(-z_1)(-z_2)\ldots(-z_{n-1}) = -1$ (equating constants) Since there are n factors : IF n is EVEN then we have $(z_0)(z_1)(z_2)\ldots(z_{n-1}) = -1$ i.e. $p = -1$ IF n is ODD then we have $-(z_0)(z_1)(z_2)\ldots(z_{n-1}) = -1$ i.e. $p = 1$
Specific behaviours
✓ expresses the LHS in the form $(z-z_0)(z-z_1)(z-z_2)\ldots(z-z_{n-1})$ ✓ states that the product of the constant terms $(-z_0)(-z_1)(-z_2)\ldots(-z_{n-1}) = -1$ ✓ states that the product depends on whether n is even or odd ✓ states the correct value for the product for each case

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