Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS SPECIALIST

Calculator-free

ATAR course examination 2022

Marking key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

Section One: Calculator-free **35% (48 Marks)**

Question 1 **(6 marks)**

Consider functions $f(x)=\sqrt{4-x}$ and $g(x)=\frac{1}{x^2}$.

(a) Determine the exact value of $g(f(-5))$. (2 marks)

Solution
$g(f(-5)) \; = \; g\left(\sqrt{4-(-5)}\right) \; = \; g(3) \; = \; \frac{1}{9}$
Specific behaviours
✓ determines $f(-5)$ correctly ✓ obtains the correct value for $g(f(-5))$

(b) Determine the domain for $f(g(x))$. (3 marks)

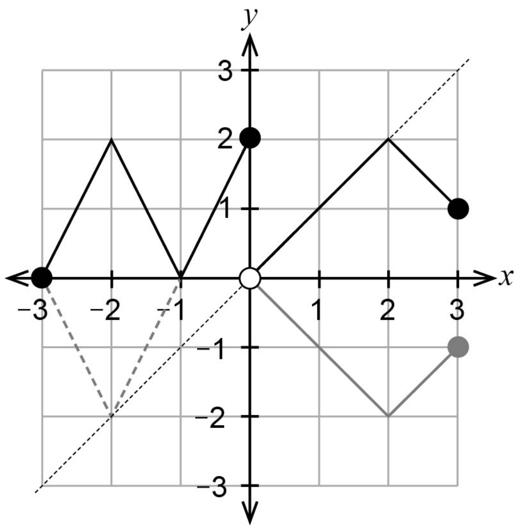
Solution
$f(g(x)) \; = \; \sqrt{4-\frac{1}{x^2}}$ This will be defined when $4-\frac{1}{x^2} \geq 0$, $x \neq 0$ since $g(x)$ must exist. i.e. $\frac{1}{x^2} \leq 4$ i.e. $x^2 \geq \frac{1}{4}$ $\therefore \; D_{fog} = \{x \mid x \geq \frac{1}{2} \cup x \leq -\frac{1}{2}\}$
Specific behaviours
✓ identifies that $f(g(x))$ is defined when $4-\frac{1}{x^2} \geq 0$ ✓ states $x \geq \frac{1}{2}$ ✓ states $x \leq -\frac{1}{2}$

(c) Explain why function g is not a one-to-one function. (1 mark)

Solution
$g(-2)=g(2)=\frac{1}{4}$ This shows that g maps two values of x to a single value. Hence g is NOT a one-to-one function BUT is a MANY-to-one function.
Specific behaviours
✓ justifies why g is not a one-to-one function

Question 2 **(7 marks)**

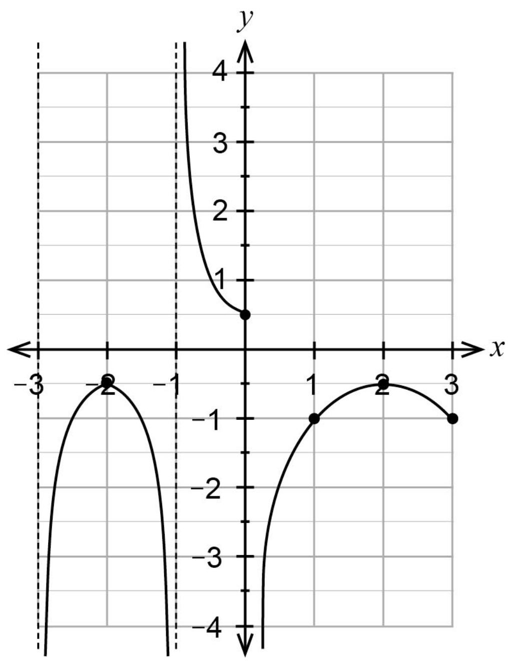
The graph of $y = f(x)$ is shown below.



(a) Solve the equation $|f(x)| = x$. (2 marks)

Solution
Equation requires the intersection between $y = \|f(x)\|$ and $y = x$. This occurs when $0 < x \le 2$.
Specific behaviours
✓ excludes $x = 0$ and includes $x = 2$ in the solution ✓ states the correct interval of real values for x

(b) Sketch the graph for $y = \dfrac{1}{f(x)}$ on the axes below. (5 marks)



Solution
See graph axes.
Specific behaviours
✓ indicates vertical asymptotes at $x = -3, -1,\ 0$ ✓ indicates correct function behaviour as $x \to -3$ and $x \to 0^+$ ✓ indicates correct function behaviour as $x \to -1$ ✓ indicates the correct curvature ✓ indicates at least one of the 5 highlighted points

Question 3 **(5 marks)**

By using one or more of the following identities:

$\cos^2 x + \sin^2 x = 1$

$\cos 2x = \cos^2 x - \sin^2 x$

$\sin 2x = 2\sin x \cos x$

evaluate exactly $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)^2 \, dx$.

Solution
$$\begin{aligned} \int_0^{\frac{\pi}{2}} (\sin x + \cos x)^2 \, dx &= \int_0^{\frac{\pi}{2}} \left(\sin^2 x + 2\sin x \cos x + \cos^2 x\right) dx \\ &= \int_0^{\frac{\pi}{2}} \left(\left(\sin^2 x + \cos^2 x\right) + 2\sin x \cos x\right) dx \\ &= \int_0^{\frac{\pi}{2}} (1 + \sin 2x) \, dx \\ &= \left[x - \frac{\cos 2x}{2}\right]_0^{\frac{\pi}{2}} \\ &= \left(\frac{\pi}{2} - \frac{\cos \pi}{2}\right) - \left(0 - \frac{\cos 0}{2}\right) \\ &= \frac{\pi}{2} + \frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= \frac{\pi}{2} + 1 \end{aligned}$$
Specific behaviours
✓ expands the integrand correctly ✓ uses the Pythagorean identity $\sin^2 x + \cos^2 = 1$ ✓ uses double angle identity for $\sin 2x$ ✓ anti-differentiates the trigonometric function correctly ✓ evaluates correctly using exact trigonometric values

Question 4 **(8 marks)**

(a) Function $f(x) = \dfrac{5(x+1)}{(x-1)(x^2+3x+1)}$ can be expressed in the form $\dfrac{a}{x-1} + \dfrac{bx+c}{x^2+3x+1}$.

Determine the value of the constants a, b and c. (3 marks)

Solution
$\dfrac{5x+5}{(x-1)(x^2+3x+1)} = \dfrac{a(x^2+3x+1)+(x-1)(bx+c)}{(x-1)(x^2+3x+1)}$ $= \dfrac{(a+b)x^2 + (3a-b+c)x + (a-c)}{(x-1)(x^2+3x+1)}$ Equating coefficients: $a+b = 0$ $3a-b+c = 5$ $a-c = 5$ Solving gives $a=2,\ b=-2,\ c=-3$ i.e. $\dfrac{5(x+1)}{(x-1)(x^2+3x+1)} = \dfrac{2}{x-1} - \dfrac{(2x+3)}{x^2+3x+1}$
Specific behaviours
✓ forms the correct expression for the equivalent numerator   ✓ equates coefficients correctly to form 3 linear equations   ✓ solves correctly to determine a, b and c

Question 4 (continued)

(b) Hence determine $\int \frac{10x+10}{(x-1)(x^2+3x+1)}\,dx$. (5 marks)

Solution
$\int \frac{10x+10}{(x-1)(x^2+3x+1)}\,dx = 2\int \frac{5x+5}{(x-1)(x^2+3x+1)}\,dx$ $= \int \frac{4}{x-1} - \frac{2(2x+3)}{x^2+3x+1}\,dx$ $= 4\ln\lvert x-1\rvert - 2\ln\lvert x^2+3x+1\rvert + k$ $= \ln\left(\frac{(x-1)^4}{(x^2+3x+1)^2}\right) + k$
Specific behaviours
✓ expresses the given integrand as double $f(x)$ ✓ writes the integrand correctly in terms of the partial fractions   ✓ anti-differentiates $\frac{a}{x-1}$ correctly using the absolute value of a natural logarithm   ✓ anti-differentiates $\frac{bx+c}{x^2+3x+1}$ correctly   ✓ uses a constant of integration

Question 5 **(6 marks)**

Consider the Cartesian equations for three planes:

$$\begin{aligned} 2x+2y+z &= 9 \\ -2x+2y-5z &= -13 \\ y-z &= -1 \end{aligned}$$

(a) Show that none of these planes is parallel to another. (2 marks)

Solution
Plane normals are $\begin{pmatrix}2\\2\\1\end{pmatrix}$, $\begin{pmatrix}-2\\2\\-5\end{pmatrix}$ and $\begin{pmatrix}0\\1\\-1\end{pmatrix}$. Since none of these normal vectors are scalar multiples of each other then the planes cannot be parallel to each other.
Specific behaviours
✓ states the normal vectors for each plane ✓ states that none of the normal vectors are multiples of each other

(b) Solve the above equations simultaneously. (3 marks)

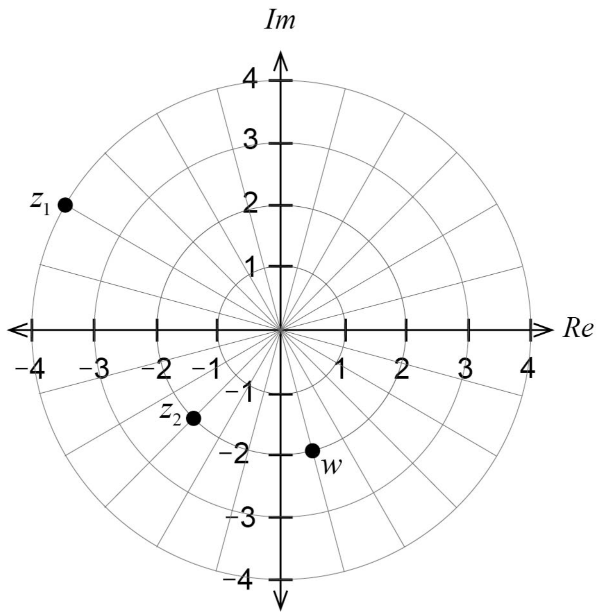
Solution
$2x+2y+z = 9 \quad \ldots (1)$ Consider $(1)+(2)$: $4y-4z = -4$ $-2x+2y-5z=-13 \quad \ldots (2)$ *i.e.* $y-z = -1 \quad \ldots (4)$ $y-z = -1 \quad \ldots (3)$ $y-z = -1 \quad \ldots (3)$ Consider $(4)-(3)$: $0=0$!! Hence there are an infinite number of solutions to these equations. Let $z=k$ where $k \in \mathbb{R}$ $\therefore\ y=k-1$ $\therefore\ 2x+2(k-1)+k=9$ *i.e.* $2x=11-3k$ $\therefore\ x=\dfrac{11-3k}{2}$ $\underset{\sim}{r} = \begin{pmatrix}\dfrac{11-3k}{2}\\k-1\\k\end{pmatrix}$ where $k \in \mathbb{R}$
Specific behaviours
✓ eliminates a variable correctly from a pair of equations ✓ states that there are an infinite number of solutions ✓ expresses correct relationships between variables

(c) State the geometric interpretation of the solution obtained in part (b). (1 mark)

Solution
The given non-parallel planes intersect in a LINE in space.
Specific behaviours
✓ states the intersection is a line in space

Question 6 **(8 marks)**

Two complex numbers $z_1 = 4cis\left(\frac{5\pi}{6}\right)$ and z_2 are shown in the Argand plane below.



(a) Determine the exact polar form for z_2. (2 marks)

Solution
$z_2 = 2\,cis\left(-\frac{3\pi}{4}\right)$ Accept also $2\,cis\left(\frac{5\pi}{4}\right)$.
Specific behaviours
✓ states the correct modulus ✓ states the correct argument

(b) Plot the complex number $w = z_1 \times (z_2)^{-1}$ on the Argand diagram above. (3 marks)

Solution
$w = \left(4cis\left(\frac{5\pi}{6}\right)\right)\times\left(2cis\left(-\frac{3\pi}{4}\right)\right)^{-1} = 4cis\left(\frac{5\pi}{6}\right)\times\frac{1}{2}cis\left(\frac{3\pi}{4}\right)$ $= 2cis\left(\frac{5\pi}{6}+\frac{3\pi}{4}\right)$ $= 2cis\left(\frac{19\pi}{12}\right)$ or $2cis\left(-\frac{5\pi}{12}\right)$ w shown on the Argand diagram above.
Specific behaviours
✓ applies DeMoivre's Theorem correctly to determine z_2^{-1} ✓ determines the correct polar form for w ✓ plots the correct position for w

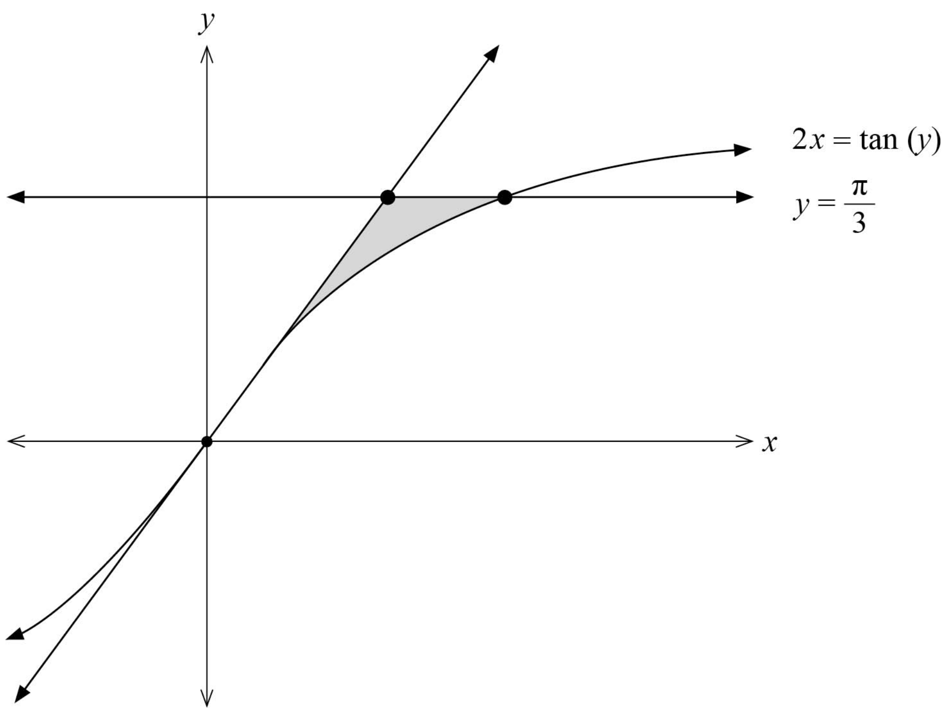
(c) If $z_1 = 4cis\left(\frac{5\pi}{6}\right)$ is a solution of the equation $z^n = r$ where r is a positive real number and n is a positive integer, determine the smallest possible value for r in the form 2^p.

Justify your answer. (3 marks)

Solution
If $z_1 = 4cis\left(\frac{5\pi}{6}\right)$ is a solution then $\left(4cis\left(\frac{5\pi}{6}\right)\right)^n = r\,cis(2\pi k)$ i.e. $2^{2n}cis\left(\frac{5n\pi}{6}\right) = r\,cis(2\pi k)$ where $k = 0, 1, 2, \ldots, n-1$ i.e. $\frac{5n}{6} = 2k$ or $n = \frac{12k}{5}$ Hence the smallest possible value of $n = 12$ (when $k = 5$) so that $n \in \mathbb{Z}^+$. $\therefore\ r = 2^{2\times 12} = 2^{24}$ is the smallest value
Specific behaviours
✓ forms the equation that determines the relationship between n and integer k ✓ deduces the smallest value for n or k ✓ states the smallest value for r as a power of 2

Question 7 **(8 marks)**

The graph of $2x = \tan(y)$ is shown along with the tangent at $x = 0$. The horizontal line $y = \frac{\pi}{3}$ is also shown.



(a) Using implicit differentiation, determine the equation of the tangent drawn at $x = 0$. (3 marks)

Solution
$\frac{d}{dx}(2x) = \frac{d}{dx}(\tan(y)) \quad \therefore \; 2 = \sec^2(y).\frac{dy}{dx}$ $\therefore \; \frac{dy}{dx} = 2\cos^2(y)$ At $y = 0, \; \frac{dy}{dx} = 2\cos^2(0) = 2(1) = 2$ Hence equation of the tangent is $y = 2x$ or $x = \frac{y}{2}$.
Specific behaviours
✓ differentiates $2x = \tan(y)$ correctly using implicit differentiation   ✓ obtains the correct expression for the derivative   ✓ determines the equation of the tangent correctly

The shaded region is bounded by the curve $2x = \tan(y)$, the tangent drawn and $y = \frac{\pi}{3}$.

(b) Write the expression for the area of the shaded region. (2 marks)

Solution
Area $= \int_0^{\frac{\pi}{3}} \left(\frac{1}{2}\tan(y) - \frac{y}{2} \right) dy$
Specific behaviours
✓ forms a definite integral using correct limits for y with correct notation ✓ forms the integrand correctly

Alternative Solution
Area $= \int_0^{\frac{\pi}{6}} \left(2x - \tan^{-1}(2x) \right) dx \; + \int_{\frac{\pi}{6}}^{\frac{\sqrt{3}}{2}} \left(\frac{\pi}{3} - \tan^{-1}(2x) \right) dx$
Specific behaviours
✓ forms two definite integrals using correct limits for x values with correct notation ✓ forms the two integrands correctly

(c) Evaluate this area exactly. (3 marks)

Solution
$Area = \int_0^{\frac{\pi}{3}} \left(\frac{1}{2}\tan(y) - \frac{y}{2} \right) dy = \int_0^{\frac{\pi}{3}} \left(\frac{1}{2}\frac{\sin y}{\cos y} - \frac{y}{2} \right) dy$ $= \left[-\frac{1}{2}\ln\lvert\cos y\rvert - \frac{y^2}{4} \right]_0^{\frac{\pi}{3}}$ $= \left[-\frac{1}{2}\ln\left(\frac{1}{2}\right) - \frac{\pi^2}{36} \right] - \left[-\frac{1}{2}\ln(1) - 0 \right]$ $= \frac{1}{2}\ln(2) - \frac{\pi^2}{36}$
Specific behaviours
✓ re-writes the tangent function in terms of sine and cosine correctly ✓ anti-differentiates correctly using the logarithm of an absolute value ✓ evaluates correctly in terms of an exact value

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