Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS METHODS

Calculator-assumed

ATAR course examination 2016

Marking Key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2016/64844

Question 9 **(7 marks)**

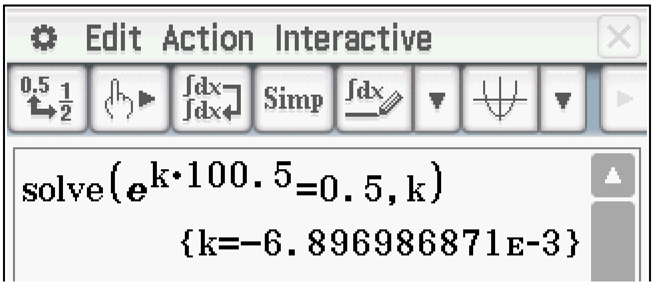
Fermium-257 is a radioactive substance whose decay rate can be modelled by the formula $P = P_o e^{kt}$, where P is the mass in grams and t is measured in days and $P_o =$ original amount and k is a constant. The time taken to decay to half of the original amount is known as half-life. The half-life of Fermium-257 is 100.5 days.

(a) Determine the value of k to three significant figures. (3 marks)

Solution

$P = P_o e^{kt}$

$e^{k100.5} = \frac{1}{2}$



$k = -0.006896$

$k = -0.00690$

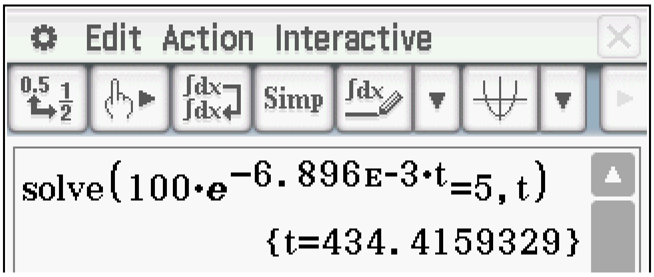
$k = -6.90 \times 10^{-3}$

Specific behaviours

✓ sets up an equation to solve for k
✓ solves for k
✓ states k to three sign figures

(b) How many days will it take for 100 grams of the substance to first decay below five grams? (2 marks)

Solution

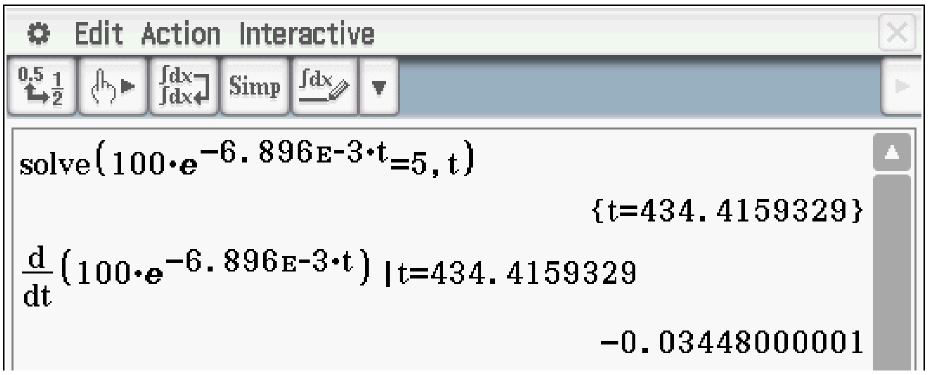


$t = 434.42$ days
Accept 434 to 435 days

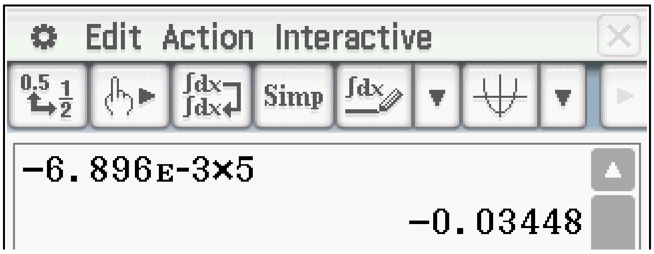
Specific behaviours

✓ sets up an equation to solve for t
✓ determines the number of days

(c) Determine the rate of change of the amount of Fermium on the day found in part (b).
(2 marks)

Solution

Specific behaviours
✓ differentiates at *t* given in part (b) ✓ determines rate of change

or

Solution

Specific behaviours
✓ multiplies k by the amount of five grams ✓ determines rate of change

Question 10 **(12 marks)**

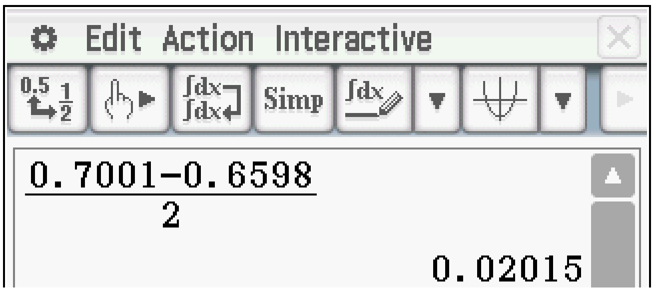
A survey in Western Australia was conducted on the popularity of a calculator known as Type A. Out of 1450 Year 12 students, the survey found that 986 students used the Type A calculator.

Determine the following.

(a) A 90% confidence interval, to three decimal places, for the proportion of Western Australian Year 12 students who use the Type A calculator. What assumption was made in calculating this interval? (3 marks)

Solution
$\hat{p} = \frac{986}{1450} = 0.68$ $s_p = \sqrt{\frac{0.68(1-0.68)}{1450}} = 0.01225$ $0.68 - 1.645(0.01225) \leq p \leq 0.68 + 1.645(0.01225)$ $0.6598 \leq \hat{p} \leq 0.7001$ $0.660 \leq \hat{p} \leq 0.700$ Assumes that sample proportions are a normal distribution.
Specific behaviours
✓ states that sample proportions form a normal distribution.   ✓ determines confidence interval   ✓ expresses interval rounded to three decimal places

(b) The margin of error in this confidence interval. (2 marks)

Solution

Specific behaviours
✓ uses confidence interval   ✓ determines margin of error

Another three surveys of Year 12 students were conducted on the use of Type A calculators across Australia.

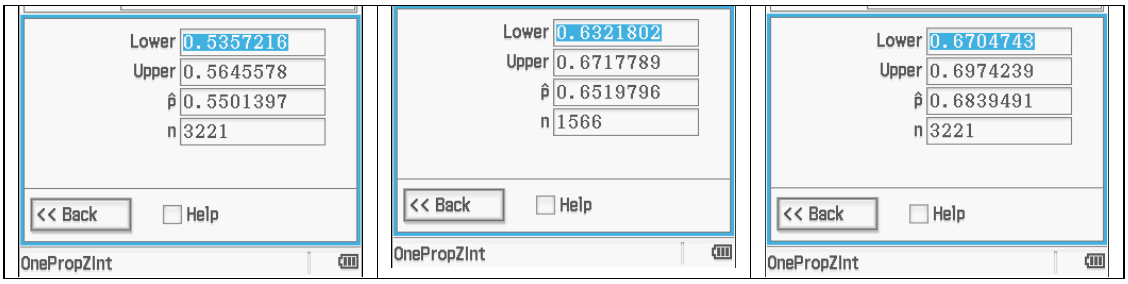
Survey 2	Survey 3	Survey 4
Type A usage 1772 out of 3221 Year 12 students	Type A usage 1021 out of 1566 Year 12 students	Type A usage 2203 out of 3221 Yr 12 students

(c) Determine which of these surveys were more likely to have been taken outside of Western Australia. Justify your answer(s). (3 marks)

Solution

Survey 2 **Survey 3** **Survey 4**

The interval for Survey 2 does not overlap with interval in part (a),



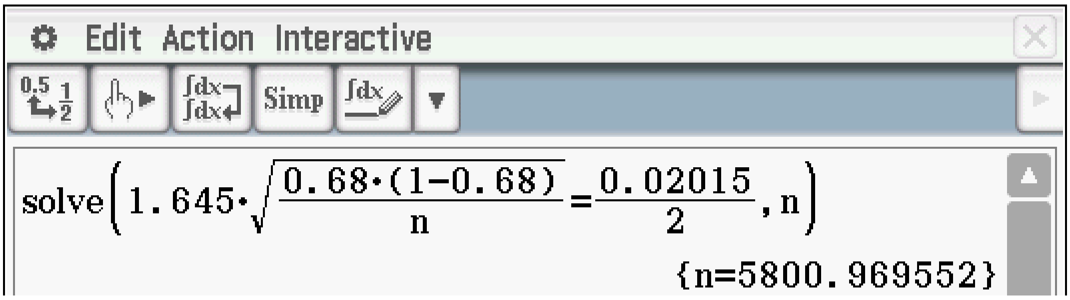
hence more likely to be taken outside of Perth.

Specific behaviours

✓ determines intervals for all three surveys
✓ compares to interval in part (a)
✓ uses an argument supported by confidence intervals

(d) Using the sample proportion of the survey at the start of the question, determine a sample size that will halve the margin of error for the proportion of Western Australian Year 12 students who use the Type A calculator, with a confidence of 90%. (4 marks)

Solution



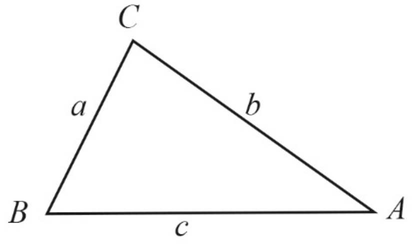
Rounds up to $n = 5801$

Specific behaviours

✓ uses the proportion from part (a)
✓ halves margin of error from part (b)
✓ sets up an equation to solve for sample size
✓ rounds sample size up to nearest integer

Question 11 **(3 marks)**

The area of a triangle can be found by the formula: $Area = \dfrac{ab\sin C}{2}$.



Using the incremental formula, determine the approximate change in area of an equilateral triangle, with each side of 10 cm, when each side increases by 0.1 cm.

Solution
$a = b = l, \quad C = \dfrac{\pi}{3}$ $A = \dfrac{1}{2} l^2 \sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{4} l^2$ $l = 10, \quad \delta l = 0.1$ $\delta A \approx \dfrac{dA}{dl}\delta l = \dfrac{2\sqrt{3}}{4}(10)0.1$ $\delta A = 0.866$ Approximate change in area of 0.866 sq cm
Specific behaviours
✓ sets up an equation for area in terms of one variable ✓ uses increments formula with correct parameters ✓ determines approximate change in area

Question 12 **(3 marks)**

The Richter magnitude, M, of an earthquake is determined from the logarithm of the amplitude, A, of waves recorded by seismographs.

$M = \log_{10}\dfrac{A}{A_o}$, where A_o is a reference value.

An earthquake in a town in New Zealand in November 2015 was estimated at 5.5 on the Richter scale, while the earthquake just north of Hayman Island measured 3.4 on the same scale. How many times larger was the amplitude of the waves in New Zealand compared to those at Hayman Island?

Solution
$M = \log_{10}\dfrac{A}{A_o}$ $A = A_o 10^M$ $\dfrac{A_{NZ}}{A_H} = \dfrac{10^{5.5}}{10^{3.4}} = 10^{2.1}$
Specific behaviours
✓ converts log statement to an index form   ✓ subtracts Richter magnitudes   ✓ determines ratio of amplitudes

Question 13 **(10 marks)**

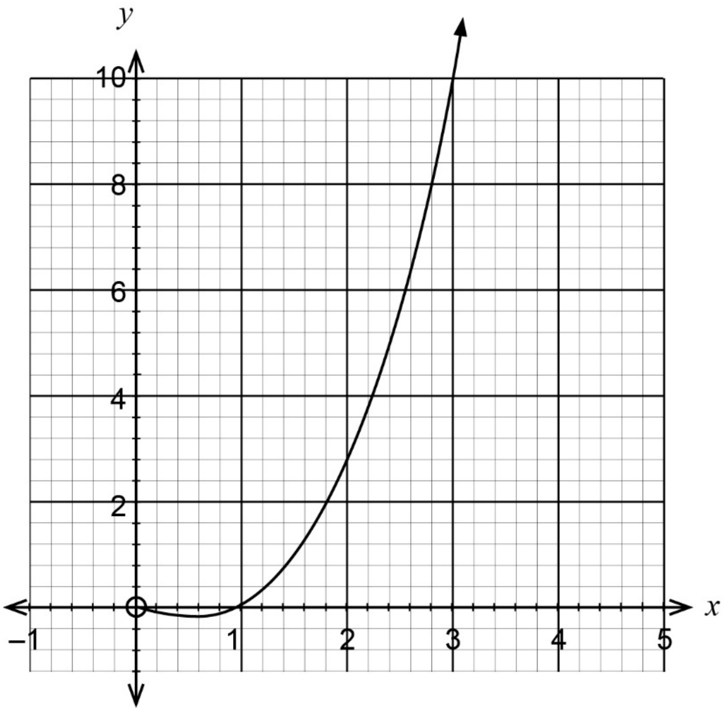
(a) Determine $\frac{d}{dx}\left(x^2 \ln x\right)$. (2 marks)

Solution
$\frac{d}{dx}\left(x^2 \ln x\right) = x^2 \frac{1}{x} + \ln x\left(2x\right)$ $= x\left(1 + 2\ln x\right)$
Specific behaviours
✓ uses product rule ✓ determines derivative

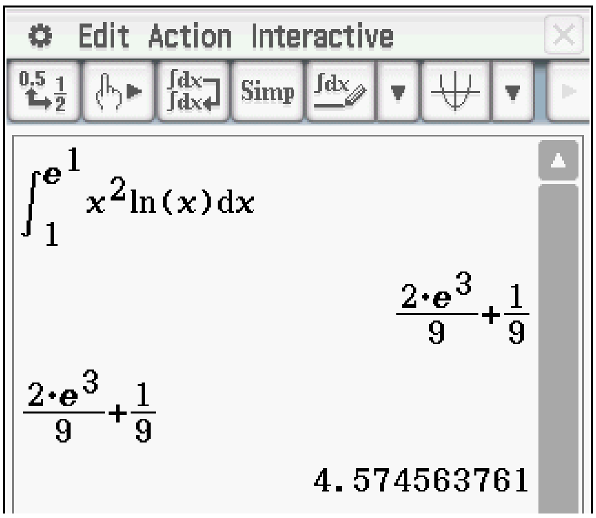
(b) Using your answer from part (a), show that the graph of $y = x^2\ 1\mathrm{n}\ x$ has only one stationary point. (3 marks)

Solution
$\frac{dy}{dx} = x\left(1 + 2\ln x\right)$ $\frac{dy}{dx} = 0, \quad \ln x = -\frac{1}{2}, \quad x \neq 0$ Only one point where derivative is zero hence only one stationary point.
Specific behaviours
✓ equates derivative to zero ✓ states that $x \neq 0$ ✓ shows that only stationary point occurs for $\ln x = -\frac{1}{2}$

(c) Sketch the graph of $y = x^2 \ln x$, showing all features. (3 marks)

Solution

Specific behaviours
✓ graphs only for $x > 0$ ✓ one turning point shown between $0 < x < 1$ ✓ open circle shown at $x = 0$

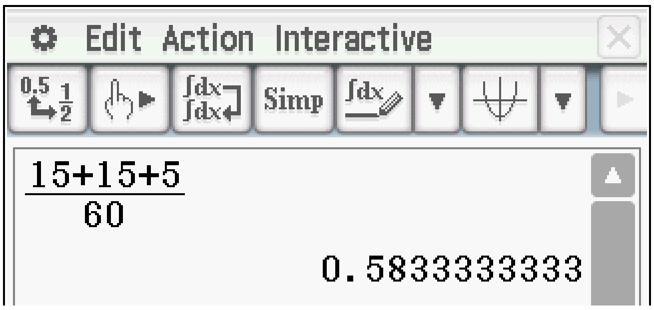
(d) Calculate the area bounded by the graph of $y = x^2 \ln x$, the x-axis, $x = 1$ and $x = e$.
(2 marks)

Solution

Specific behaviours
✓ expresses area as a definite integral ✓ determines area

Question 14 **(9 marks)**

The simulation of a loaded (unfair) five-sided die rolled 60 times is recorded with the following results.

(a) Calculate the proportion of prime numbers recorded in this simulation. (2 marks)

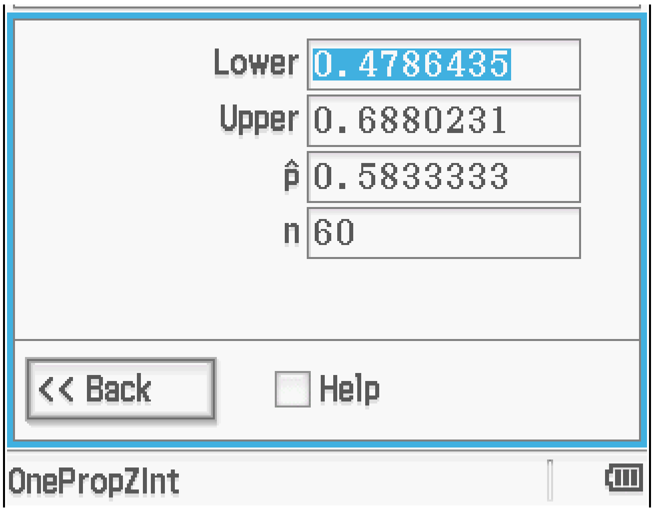
Solution

Specific behaviours
✓ identifies the prime numbers ✓ determines proportion

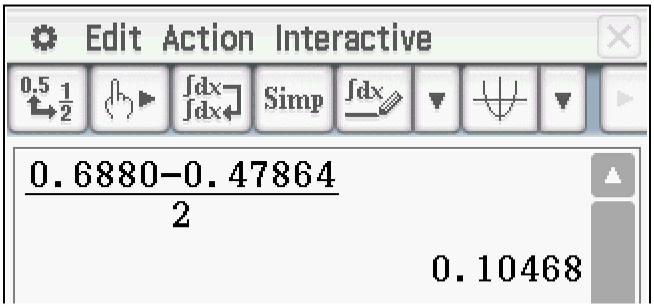
(b) Determine the mean and standard deviation for the sample proportion of prime numbers in 60 tosses, using the results above. (2 marks)

Solution
$\hat{p} = 0.58$ $s_x = \sqrt{\dfrac{0.58(1-0.58)}{60}} = 0.0637$
Specific behaviours
✓ determines the mean ✓ determines standard deviation

(c) It has been decided to create a confidence interval for the proportion of prime numbers using the simulation results on page 8. The level of confidence will be chosen from 90% or 95%. Explain which level of confidence will give the smallest margin of error. State this margin of error. (3 marks)

Solution
Smallest margin of error occurs for smallest confidence percentage 90%. There is a trade-off between level of confidence and margin of error.





Specific behaviours
✓ uses 90% confidence ✓ states trade-off between confidence and margin of error ✓ determines margin of error

This simulation of 60 rolls of the die is performed another 200 times, with the proportion of prime numbers recorded each time and graphed.

(d) Comment briefly on the key features of this graph. (2 marks)

Solution
Graph takes the shape of a binomial distribution. Approaches the shape of a normal distribution for large values of n. Distribution is centred on 0.58.
Specific behaviours
✓ at least one of the descriptors above ✓ at least two descriptors above

Question 15 **(6 marks)**

A tetrahedral die has the numbers 1 to 4 on each face. When thrown, each side is equally likely to land facedown. Let X be defined as the sum of the numbers on the facedown side when the die is thrown twice.

(a) Complete the following table. (1 mark)

Solution

		Roll two			
	Sum of two rolls	1	2	3	4
Roll one	1	1 + 1 = 2	3	**4**	**5**
	2	3	**4**	**5**	**6**
	3	**4**	5	**6**	**7**
	4	**5**	**6**	**7**	**8**

Specific behaviours

✓ enters all missing totals in table

(b) (i) Hence, or otherwise, complete the probability distribution of X, which is given by the following table. (1 mark)

Solution

x	2	3	4	5	6	7	8
$P(X = x)$	$\frac{1}{16}$	$\mathbf{\frac{2}{16}}$	$\mathbf{\frac{3}{16}}$	$\mathbf{\frac{4}{16}}$	$\mathbf{\frac{3}{16}}$	$\mathbf{\frac{2}{16}}$	$\frac{1}{16}$

Specific behaviours

✓ completes table

(ii) Calculate the probability of obtaining a sum of five or less. (2 marks)

Solution

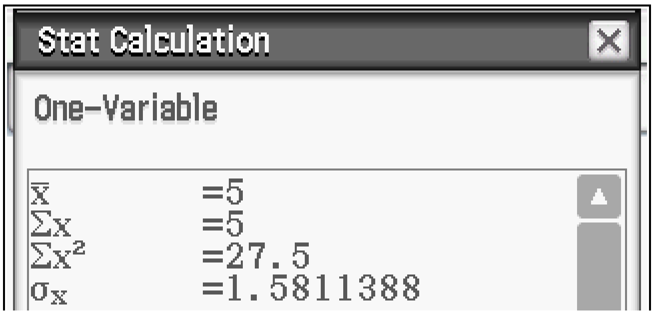
$$P(S \leq 5) = \frac{4+3+2+1}{16} = \frac{10}{16}$$

Specific behaviours

✓ uses all allowed values of sums
✓ determines probability

(iii) Determine the mean and standard deviation for X. (2 marks)

Solution



Mean = 5
Standard deviation=1.58

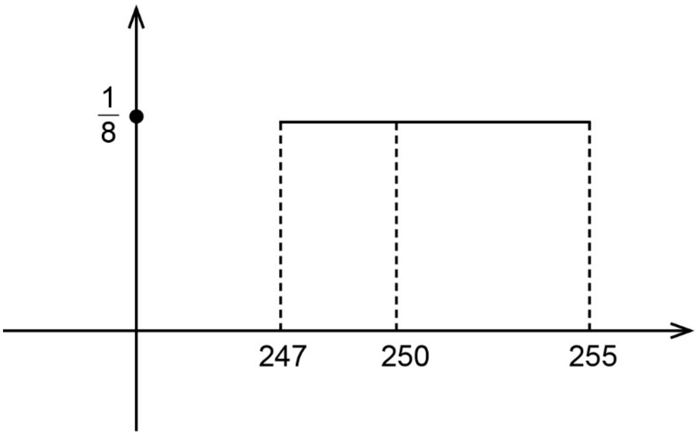
Specific behaviours

✓ determines mean
✓ determines standard deviation

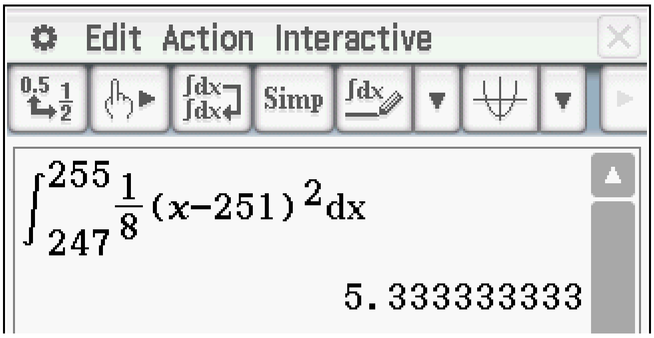
Question 16 **(10 marks)**

An automated milk bottling machine fills bottles uniformly to between 247 ml and 255 ml. The label on the bottle states that it holds 250 ml.

(a) Determine the probability that a bottle selected randomly from the conveyor belt of this machine contains less than the labelled amount. (3 marks)

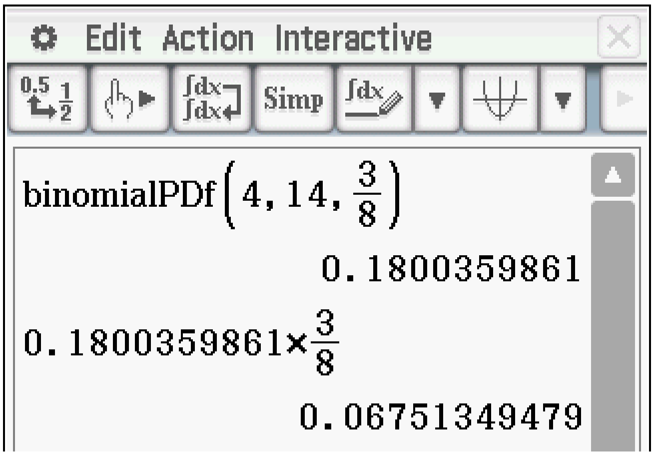
Solution
 $P(X < 250) = \frac{3}{8}$
Specific behaviours
✓ evaluates pdf value ✓ uses area to calculate probability ✓ determines probability

(b) Calculate the mean and standard deviation of the amount of milk in the bottles. (4 marks)

Solution
Mean = $\frac{255+247}{2} = 251$  Standard deviation = $\sqrt{Var(x)}$ = 2.3094
Specific behaviours
✓ determines mean ✓ uses appropriate pdf function ✓ sets up an integral to find variance ✓ determines standard deviation

A worker selects bottles from the conveyor belt, one at a time.

(c) Determine the probability that it takes the selection of 15 bottles before five bottles containing less than the labelled amount have been selected. (3 marks)

Solution

Specific behaviours
✓ uses binomial distribution for first 14 selections ✓ determines probability for the first four bottles ✓ determines final probability

Question 17 **(7 marks)**

A school has analysed the examination scores for all its Year 12 students taking Methods as a subject. Let X = the examination percentage scores of all the Methods Year 12 students at the school. The school found that the mean was 75 with a standard deviation of 22.

Determine the following.

(a) $E(X+5)$ (1 mark)

Solution
$E(X+5)=E(X)+5=80$
Specific behaviours
✓ determines mean

(b) $Var(25-2X)$ (2 marks)

Solution
$Var(25-2X)=2^2Var(X)=4\times 22\times 22=1936$
Specific behaviours
✓ uses a positive factor of four ✓ determines variance

The school has decided to scale the results using the transformation $Y=aX+b$ where a and b are constants and Y = the scaled percentage scores. The aim is to change the mean to 60 and the standard deviation to 15.

(c) Determine the values of a and b. (4 marks)

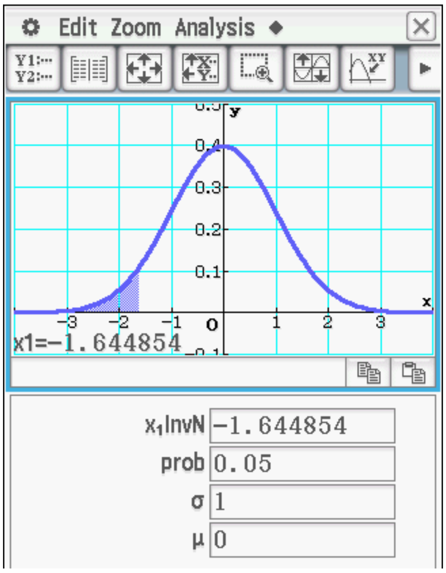
Solution
$15=22a$ $a=\frac{15}{22}\approx 0.682$ $60=75a+b$ $b=\frac{195}{22}\approx 8.864$
Specific behaviours
✓ determines change on standard deviation first ✓ sets up at least one equations for a and b ✓ determines a ✓ determines b

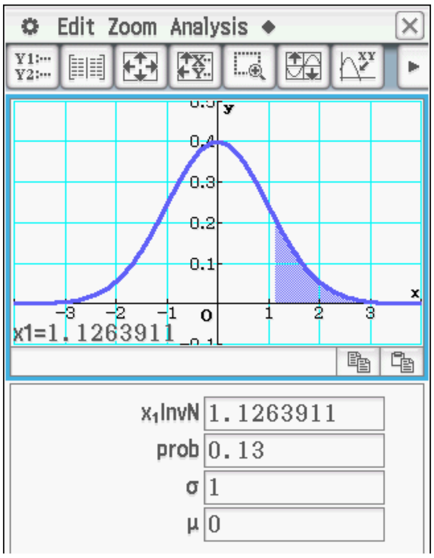
Question 18 **(6 marks)**

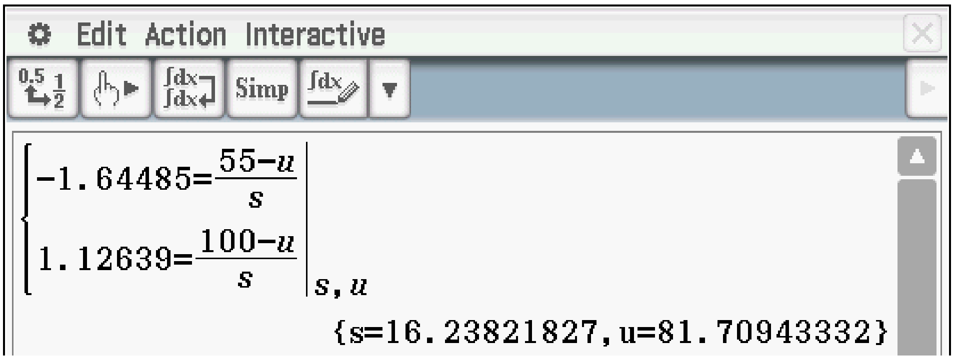
The waiting times at a Perth Airport departure lounge have been found to be normally distributed. It is observed that passengers wait for less than 55 minutes, 5% of the time, while there is a 13% chance that the waiting times will be greater than 100 minutes.

(a) Determine the mean and standard deviation for the waiting times at Perth Airport departure lounge. (5 marks)

Solution



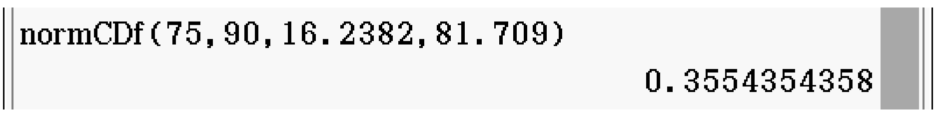




Specific behaviours
✓ determines both z scores
✓ sets up at least one equation with mean and standard deviation
✓ sets up two equations with mean and standard deviation
✓ solves for mean
✓ solves for standard deviation

(b) Determine the probability that the waiting time will be between 75 and 90 minutes. (1 mark)

Solution



Specific behaviours
✓ determines probability

Question 19 **(8 marks)**

The displacement in centimetres of a particle from the point O in a straight line is given by $x(t) = \frac{1}{3}\left(\frac{t}{2} - 4\right)^2 - 2$ for $0 \le t \le 10$, where t is measured in seconds.

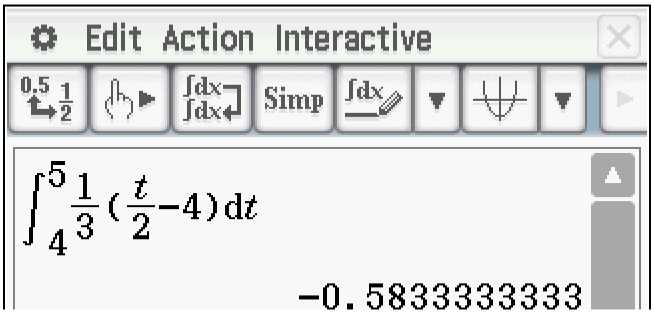
Calculate the:

(a) time(s) that the particle is at rest. (2 marks)

Solution
$\frac{dx}{dt} = \frac{1}{3}\left(\frac{t}{2} - 4\right) = 0$ $\frac{t}{2} = 4$ $t = 8$
Specific behaviours
✓ differentiates to determine velocity ✓ solves for time that velocity equals zero

(b) displacement of the particle during the fifth second. (2 marks)

Solution



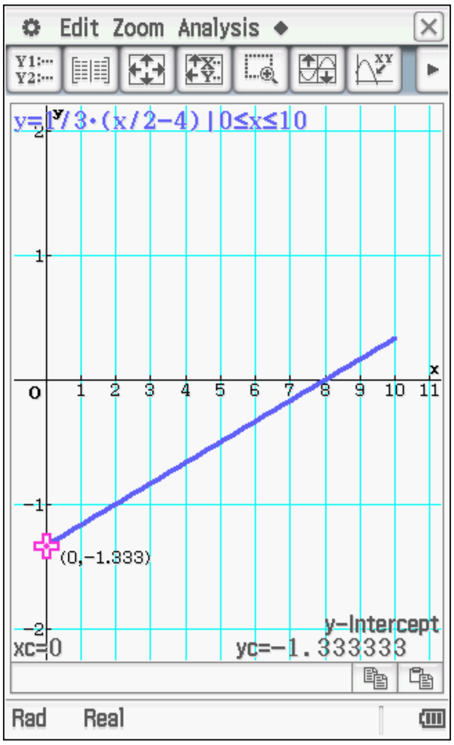
Displacement = -0.5833cm

Specific behaviours
✓ examines motion between $t = 4$ and $t = 5$ ✓ determines change in displacement

(c) maximum speed of the particle and the time when this occurs. (2 marks)

Solution

$$\frac{dx}{dt}=\frac{1}{3}\left(\frac{t}{2}-4\right) \quad ,0\le t\le 10$$

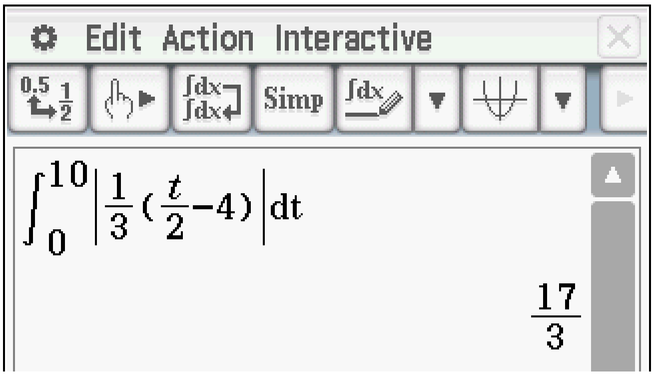


Maximum speed $\frac{4}{3}cm/s \quad ,t=0$

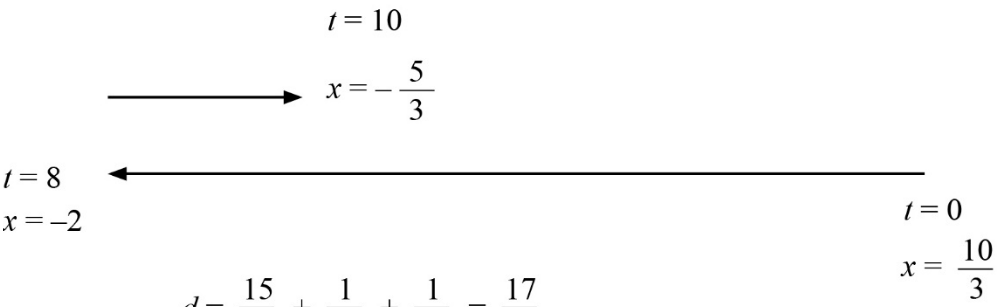
Specific behaviours

✓ examines velocity at endpoints $t=0, 10$ seconds
✓ determines maximum speed

(d) total distance travelled in the first 10 seconds. (2 marks)

Solution

Specific behaviours
✓ sets up an integral to determine distance travelled ✓ determines distance travelled

or

Solution
 $d = \frac{15}{3} + \frac{1}{3} + \frac{1}{3} = \frac{17}{3}$
Specific behaviours
✓ sets up a pathway of motion in first 10 seconds ✓ determines distance travelled

Question 20 **(14 marks)**

A chocolate factory produces chocolates of which 80% are pink. Each box of chocolates contains exactly 30 pieces.

(a) Identify the probability distribution of X = the number of pink chocolates in a single box and also give the mean and standard deviation. (3 marks)

Solution

$$X \approx Bin\left(30, \frac{4}{5}\right)$$

$$u = 24$$

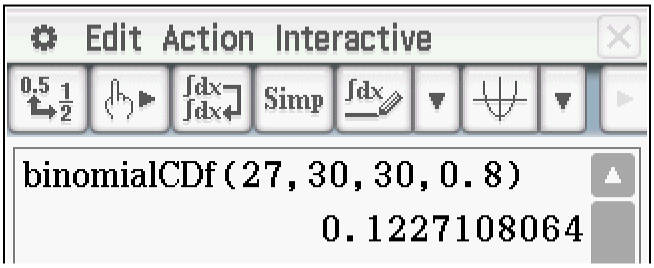
$$s = \sqrt{30\frac{4}{5}\left(1-\frac{4}{5}\right)} = 2.191$$

Specific behaviours

✓ identifies binomial distribution
✓ determines mean
✓ determines standard deviation

(b) Determine the probability, to three decimal places, that there are at least 27 pink chocolates in a randomly selected box. (3 marks)

Solution



0.123 (rounded)

Specific behaviours

✓ uses binomial distribution with correct parameters
✓ determines probability
✓ rounds to three decimal places

Quality Control collects samples sizes of 20 boxes and counts the number of pink chocolates in total.

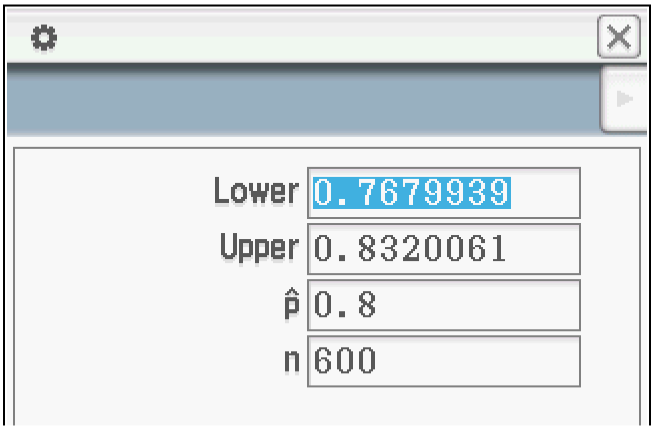
(c) Determine a 95% confidence interval for the proportion of pink chocolates in a sample of 20 boxes, using the assumption that 80% of chocolates in the sample are pink. (2 marks)

Solution

$n = 20 \times 30 = 600$

$p = 0.8$

$x = 0.8 \times 600 = 480$



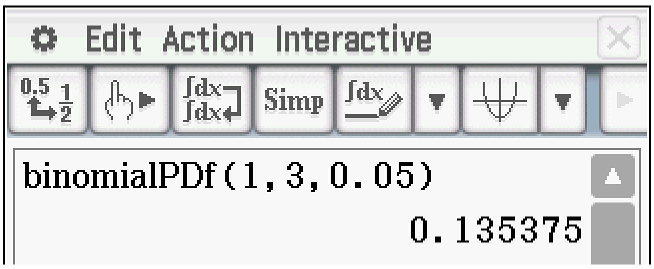
95% confidence interval $0.768 \le p \le 0.832$

Specific behaviours

✓ determines correct values for n, p and x
✓ determines confidence interval

(d) Quality Control collects three samples and determines a 95% confidence interval each time. Determine the probability that only one of these intervals will not contain the true value 0.8 of the proportion of pink chocolates (2 marks)

Solution

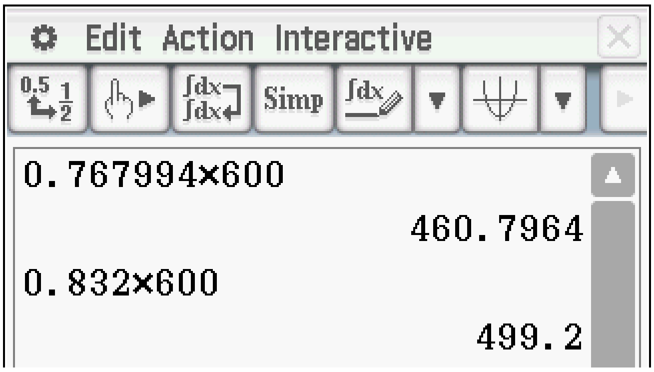


Specific behaviours

✓ uses a binomial distribution
✓ determines probability

(e) Using your 95% confidence interval in part (c), determine the range in which the expected number of pink chocolates in a sample of 20 boxes would lie. (2 marks)

Solution



The range would lie between 460 and 499 pink chocolates.

Specific behaviours

✓ uses confidence interval
✓ multiplies proportion by total number of chocolates in sample

Quality Control counted the number of pink chocolates in five samples as shown below.

Sample	**1**	**2**	**3**	**4**	**5**
Number of pink chocolates	433	463	482	473	566

(f) Decide which samples lie outside the 95% confidence interval, if any. Justify. (2 marks)

Solution

Samples 1 and 5 lie outside the range in part (e), hence lie outside proportion interval.

Specific behaviours

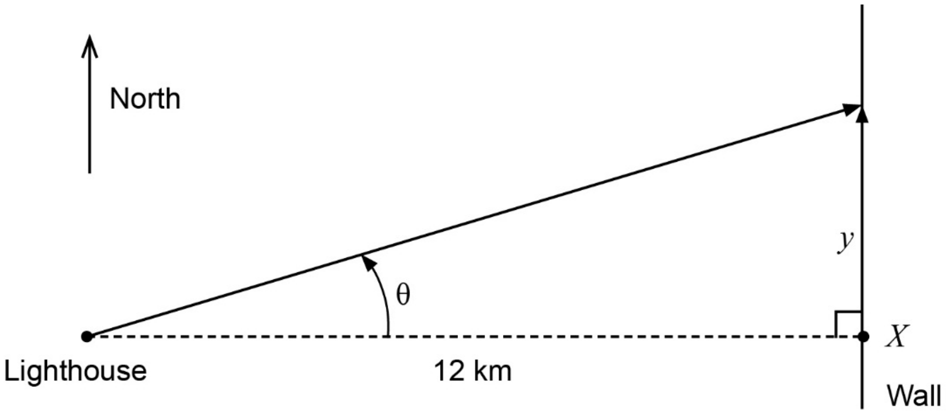
✓ uses range of chocolates from part (e)
✓ presents an argument using confidence intervals

Question 21 **(6 marks)**

A lighthouse is situated 12 km away from the shoreline, opposite point X as seen in the diagram below. A long brick wall is placed along the shoreline and at night the light from the lighthouse can be seen moving along this wall.

Let y = displacement of light on the wall from point X and θ = angle of the rotating light from the lighthouse.

The light is revolving anticlockwise at a uniform rate of three revolutions per minute ($\frac{d\theta}{dt} = 6\pi$ radians/minute).



(a) Show that $\frac{dy}{d\theta} = \frac{12}{\cos^2\theta}$. (3 marks)

Solution
$\tan\theta = \frac{y}{12}$ $y = 12\tan\theta$ $y = \frac{12\sin\theta}{\cos\theta}$ $\frac{dy}{d\theta} = \frac{12\cos\theta\cos\theta + 12\sin\theta\sin\theta}{\cos^2\theta}$ $\frac{dy}{d\theta} = \frac{12}{\cos^2\theta}$
Specific behaviours
✓ express y in terms of $\tan\theta$ ✓ differentiates $\tan\theta$ ✓ expresses as $\frac{12}{\cos^2\theta}$ equivalent

(b) Determine the velocity, in kilometres per minute, of the light on the wall when the light is 5 km north of point X. (3 marks)

(Hint: $\dfrac{dy}{dt}=\dfrac{dy}{d\theta}\times\dfrac{d\theta}{dt}$)

Solution
$\dfrac{dy}{dt}=\dfrac{dy}{d\theta}\dfrac{d\theta}{dt}$ $=\dfrac{12}{\cos^2\theta}6\pi=72\pi\left(\cos\theta\right)^{-2}$ When $x=5$ $\tan\theta=\dfrac{5}{12}$ so that $\cos\theta=\dfrac{12}{13}$ $\theta\approx 22.62°$ (0.395 radians) $\dfrac{dy}{dt}=72\pi\left(\cos\theta\right)^{-2}$ $=\dfrac{72\pi}{12^2}13^2$ $=\dfrac{169}{2}\pi$ ≈ 265.465 Velocity = 265.465 kilometres per minute
Specific behaviours
✓ determines $\cos\theta$ for $x=5$ ✓ uses chain rule with $\dfrac{d\theta}{dt}=6\pi$ ✓ determines velocity

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