Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS SPECIALIST

Calculator-assumed

ATAR course examination 2016

Marking Key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2016/64834

Section Two: Calculator-assumed **65% (97 Marks)**

Question 9 **(5 marks)**

Consider the integral $I = \int x\sqrt{(1+x)^n}\ dx$, where n is any positive integer.

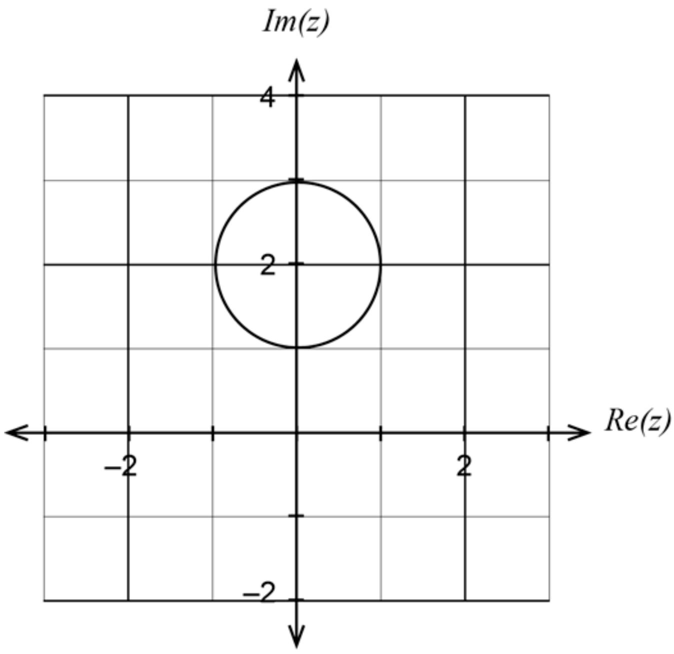
Using the substitution $u = 1 + x$ and an appropriate anti-derivative, develop a simplified expression for I in terms of x and n.

Solution
$I = \int x\sqrt{(1+x)^n}\ dx \qquad u = 1 + x \qquad \therefore\ dx = du$ $= \int (u-1)\ u^{\frac{n}{2}}.\ du$ $= \int \left(u^{\frac{n}{2}+1} - u^{\frac{n}{2}} \right) du$ $= \dfrac{u^{\frac{n}{2}+2}}{\frac{n}{2}+2} - \dfrac{u^{\frac{n}{2}+1}}{\frac{n}{2}+1} + c$ $= \dfrac{u^{\frac{n+4}{2}}}{\frac{n+4}{2}} - \dfrac{u^{\frac{n+2}{2}}}{\frac{n+2}{2}} + c$ $= \dfrac{2(1+x)^{\frac{n+4}{2}}}{n+4} - \dfrac{2(1+x)^{\frac{n+2}{2}}}{n+2} + c = \dfrac{2\sqrt{(1+x)^{n+4}}}{n+4} - \dfrac{2\sqrt{(1+x)^{n+2}}}{n+2} + c$ **or** $= \dfrac{2(nx+2x-2)(1+x)^{\frac{n}{2}+1}}{(n+4)(n+2)}$
Specific behaviours
✓ writes the integral in terms of u correctly ✓ multiplies the integrand to obtain the correct powers of u ✓ anti-differentiates correctly ✓ re-writes the anti-derivative in terms of x ✓ simplifies the expression appropriately

Question 10 **(9 marks)**

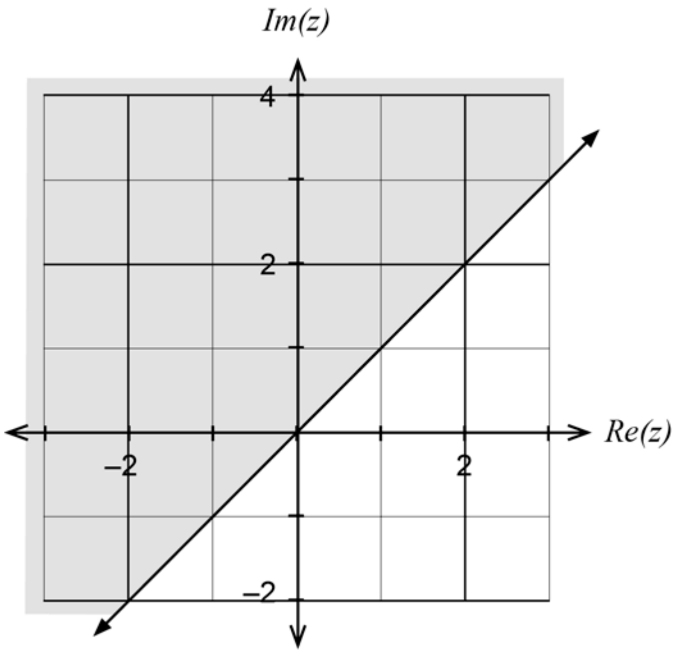
On the Argand planes below sketch the locus of the complex number $z = x + iy$ given by:

(a) $|z - 2i| = 1$. (3 marks)



Solution
Require distance of z from $(0,2)$ to be equal to one unit i.e. a circle of radius one unit with centre $(0,2)$
Specific behaviours
✓ indicates a circle as the locus ✓ indicates the correct radius ✓ indicates the correct centre

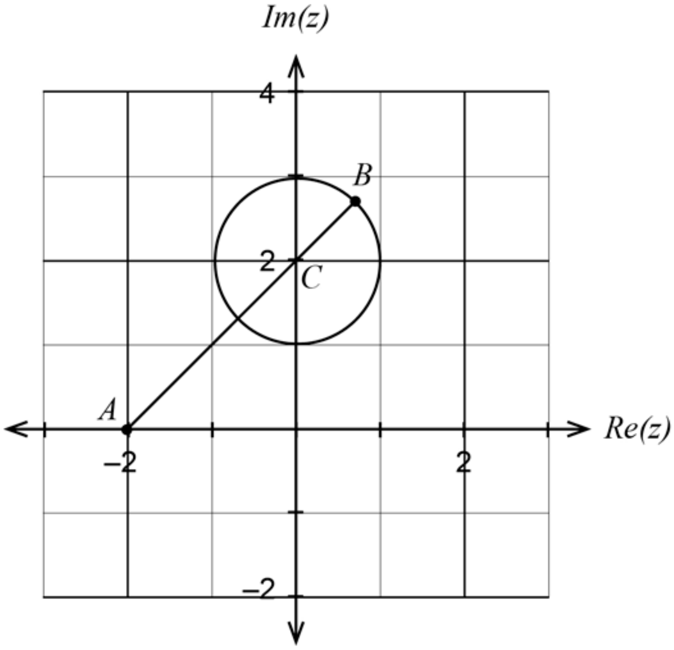
(b) $|z-1+i| \geq |z+1-i|$. (3 marks)



Solution
Require distance of z from $(1,-1)$ to be equal or greater than distance from $(-1,1)$
Specific behaviours
✓ indicates the points $(1,-1)$ and $(-1,1)$ correctly ✓ indicates the boundary position $y = x$ correctly ✓ indicates the correct half plane (including the boundary)

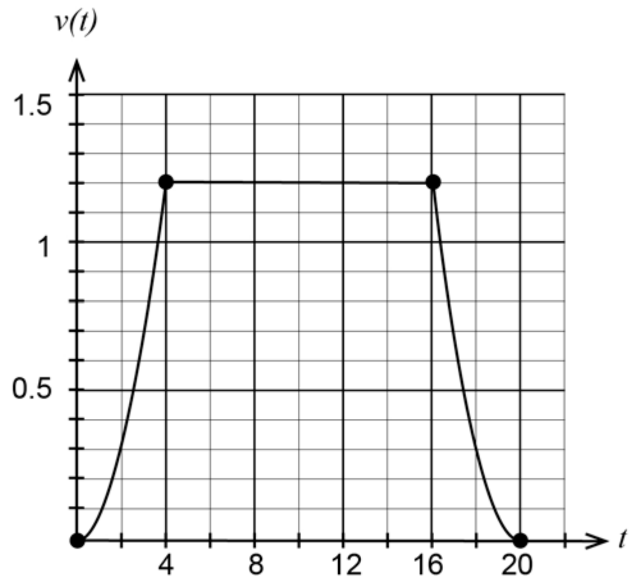
(c) For the locus $|z-2i|=1$ from part (a), state the exact maximum value for $|z+2|$.

(3 marks)

Solution
Require the maximum of $|z+2|=|z-(-2)|=$ distance of z from -2. This maximum length AB is shown below. $AB = AC + CB$ $= \sqrt{2^2+2^2}+1$ $= \sqrt{8}+1$  Hence the maximum value for $|z+2| = 2\sqrt{2}+1$ units (approximately 3.828 units)
Specific behaviours
✓ indicates the vector/line segment that gives the maximum value for $|z+2|$ ✓ forms the correct expression for the maximum ✓ states the correct exact value

Question 11 **(7 marks)**

A lift goes up within a high rise building so that its velocity $v(t)$ is given by the graph shown below. The maximum velocity of the lift during its ascent is 1.2 ms^{-1}. For the first four seconds, the acceleration is given by $a(t) = kt$. For the final four seconds of its ascent, the lift decelerates at the same rate.



(a) Show that the value of the constant $k = \dfrac{3}{20}$. (2 marks)

Solution
For $0 \le t \le 4$ $\quad a(t) = kt$ so $v(t) = \dfrac{kt^2}{2} + c$ Since $v(0) = 0$ then $c = 0$. Given $v(4) = 1.2$, $\quad 1.2 = \dfrac{k(4)^2}{2}$ $\quad \therefore \quad k = \dfrac{3}{20} = 0.15$ i.e. $v(t) = 0.075t^2$
Specific behaviours
✓ anti-differentiates the acceleration function correctly ✓ uses or states that $v(0) = 0$ and $v(4) = 1.2$ to determine the value of k

(b) Using the incremental formula, determine the approximate change in velocity v from $t = 2$ to $t = 2.1$ seconds. (2 marks)

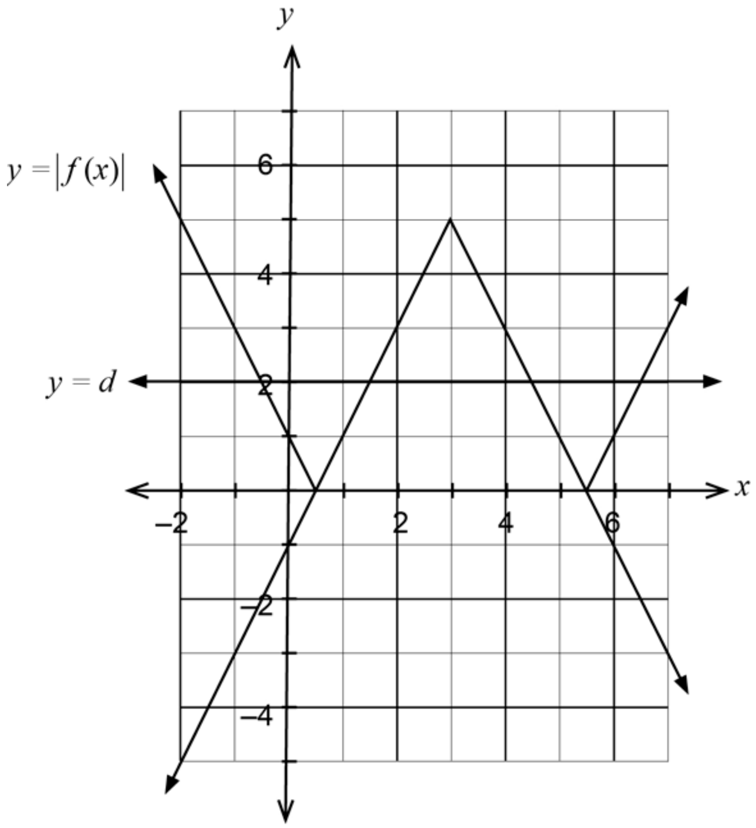
Solution
Using $\Delta v \approx \frac{dv}{dt} \times \Delta t$ $= a(2) \times (0.1)$ $= \frac{3}{20}(2) \times (0.1)$ $= 0.03 \text{ ms}^{-1}$
Specific behaviours
✓ applies the incremental formula for Δv correctly using $a(2)$ ✓ calculates the change in velocity correctly

(c) Determine the total distance that the lift travels upwards during its ascent, correct to the nearest 0.1 m. (3 marks)

Solution
Total distance travelled = **Area** under the $v(t)$ graph from $t = 0$ to $t = 20$ $= 2 \times \int_0^4 v(t)\,dt + (12)(1.2)$ $= 2 \times (1.6) + 14.4 = 17.6$ metres
Specific behaviours
✓ states that the distance required is the area under the velocity time graph ✓ writes the correct expression for the distance using an integral of velocity ✓ evaluates correctly

Question 12 **(6 marks)**

The graph of $f(x) = a|x-b|+c$ is shown below.



(a) Determine the values for the constants a, b and c. (3 marks)

Solution
From the bounce point $(3,5)$ $b=3,\ c=5$ Slopes of each line are $m=-2$ and 2 $\therefore$ $a=-2$ since the graph is inverted i.e. $f(x) = -2\|x-3\|+5$
Specific behaviours
✓ states the value of a correctly ✓ states the value of b correctly ✓ states the value of c correctly

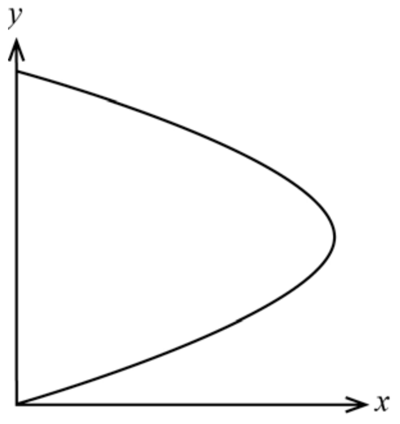
Consider the equation $\left|f(x)\right| = d$.

(b) If the equation $\left|f(x)\right| = d$ has exactly four solutions, state the possible value(s) for the constant d. Explain. (3 marks)

Solution
See above for graph of $y = \left\|f(x)\right\|$. For four solutions to $\left\|f(x)\right\| = d$ there needs to be four points of intersection of $y = \left\|f(x)\right\|$ with $y = d$. This occurs when $0 < d < 5$.
Specific behaviours
✓ indicates that $d = 0$ and $d = 5$ are significant values ✓ writes the correct interval of values for d ✓ explains/indicates that there will be four points of intersection with $y = \left\|f(x)\right\|$

Question 13 **(5 marks)**

The graph of the curve $2x = \sin(y)$ is sketched for $0 \le y \le \pi$.



(a) Determine the expression for $\frac{dy}{dx}$ in terms of y. (2 marks)

Solution
$\frac{d}{dx}(2x) = \frac{d}{dx}(\sin y)$ i.e. $2 = \cos y \cdot \frac{dy}{dx} \quad \therefore \frac{dy}{dx} = \frac{2}{\cos y}$
Specific behaviours
✓ uses implicit differentiation correctly   ✓ obtains the derivative in terms of y

(b) Determine the area of the region bounded by the curve $2x = \sin(y)$ and the y axis. (3 marks)

Solution
Area $= \int_0^{\pi} x.dy = \int_0^{\pi} \left(\frac{1}{2}\sin y\right) dy$ $= 1$ square unit (using CAS)
Specific behaviours
✓ writes a definite integral using the correct limits   ✓ writes the integrand correctly   ✓ evaluates the area correctly

Question 14 **(7 marks)**

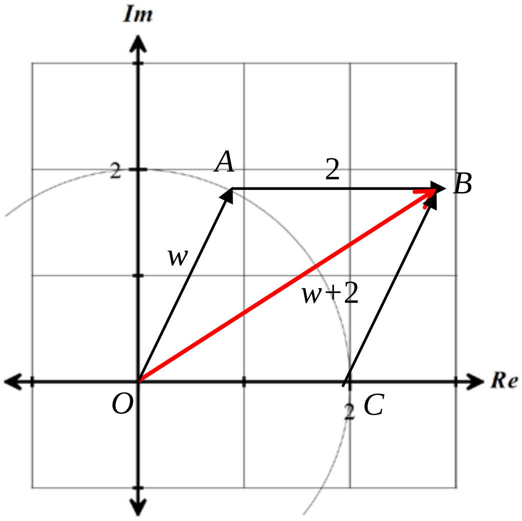
Consider the complex equation $z^4 = -16i$.

(a) Solve the equation giving all solutions in the form $r\,\text{cis}\,\theta$ where $-\pi < \theta \le \pi$. (4 marks)

Solution
$z^4 = -16i = 16\,\text{cis}\left(-\frac{\pi}{2}\right)$ Solutions are : $z = 16^{\frac{1}{4}} cis\frac{1}{4}\left(-\frac{\pi}{2}+2\pi k\right) = 2cis\left(-\frac{\pi}{8}+\frac{\pi}{2}k\right)$ where $k = 0,1,2,3$ i.e. $z_0 = 2cis\left(-\frac{\pi}{8}\right) = 2cis\left(-22.5^o\right)$, $z_1 = 2cis\left(-\frac{\pi}{8}+\frac{\pi}{2}\right) = 2cis\left(\frac{3\pi}{8}\right) = 2cis\left(67.5^o\right)$ $z_2 = 2cis\left(\frac{7\pi}{8}\right) = 2cis\left(157.5^o\right)$, $z_3 = 2cis\left(-\frac{5\pi}{8}\right) = 2cis\left(-112.5^o\right)$
Specific behaviours
✓ converts $-16i$ to polar form correctly   ✓ forms the correct expression for the roots using De Moivre's Theorem   ✓ states one root correctly   ✓ states all 4 roots correctly

Let w be the solution to $z^4 = -16i$ that has the least positive argument.

(b) Determine the value for $arg\left(w+2\right)$. (3 marks)

Solution
$w = 2cis\left(\frac{3\pi}{8}\right)$ Consider $w+2$ being represented by $\overrightarrow{OB}$ in the Argand plane.

ΔOAB is isosceles as $OA=AB=2$. $s\angle OAB = 90+22.5^0 = 112.5^o = \frac{5\pi}{8}$ $\therefore\ s\angle AOB = \frac{1}{2}\left(\frac{3\pi}{8}\right) = \frac{3\pi}{16}$ $\therefore\ arg\left(w+2\right) = \frac{3\pi}{16} = 33.75^0 = 0.589...$ OR as $OABC$ is a rhombus,   then $\overrightarrow{OB}$ bisects $\angle AOC$. $\therefore\ arg\left(w+2\right) = \frac{1}{2}arg\left(w\right) = \frac{3\pi}{16}$
Specific behaviours
✓ recognises $w = 2cis\left(\frac{3\pi}{8}\right)$ as having the least positive argument **or** $w = 2cis\left(\frac{3\pi}{8}\right)$ as a vector in the Argand plane   ✓ applies a geometric property correctly   ✓ states the correct argument for $arg\left(w+2\right)$

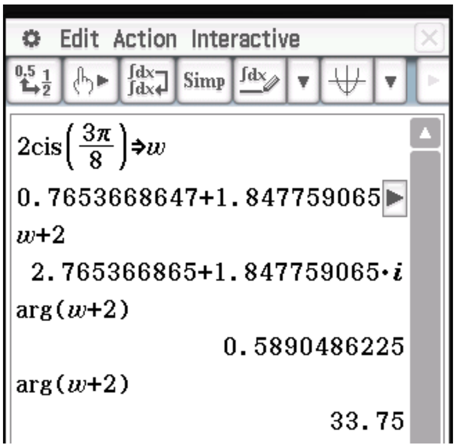
Alternative solution

$w = 2cis\left(\frac{3\pi}{8}\right) = 0.765 + 1.848i$ has the least positive argument.

$\therefore \quad arg(w) = 1.178... = 67.5^o$

$w + 2 = 2 + 2cis\left(\frac{3\pi}{8}\right) = 2.765 + 1.848i$

Using CAS : $\therefore \quad arg(w+2) = 33.75^o$ or $0.589...$ or $\frac{3\pi}{16}$



Specific behaviours

✓ recognises $w = 2cis\left(\frac{3\pi}{8}\right)$ as having the least positive argument

✓ determines $(w+2)$ as a complex number

✓ states the correct argument for $arg(w+2)$

If a candidate considers the ‘least positive’ to mean the ‘MOST NEGATIVE’:

Solution using MOST NEGATIVE interpretation

$w = 2cis\left(-\frac{5\pi}{8}\right) \; = \; -0.765 - 1.848i$ has the MOST NEGATIVE argument.

$\therefore \quad arg(w) \; = \; -1.963... \; = \; -112.5^o$

$w + 2 \; = \; 2 + 2cis\left(-\frac{5\pi}{8}\right) \; = \; 1.235 - 1.848i$

Using CAS : $\therefore \quad arg(w+2) \; = \; -56.25^o$ or $-0.982...$ or $-\frac{5\pi}{16}$

$2\text{cis}\left(\frac{-5\pi}{8}\right) \Rightarrow w$

```
-0.7653668647-1.84775906▶
w+2
 1.234633135-1.847759065•i
arg(w+2)
                -0.9817477042
arg(w+2)
                       -56.25
```

Specific behaviours

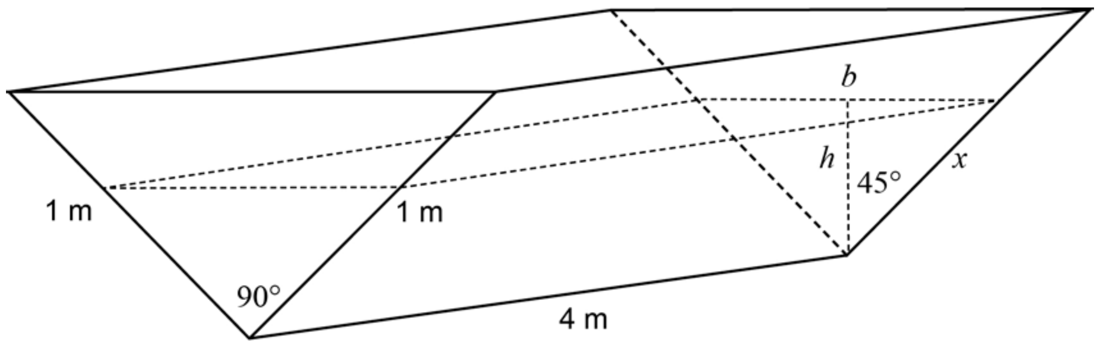
✓ recognises $w = 2cis\left(-\frac{5\pi}{8}\right)$ as having the MOST NEGATIVE argument

✓ determines $(w+2)$ as a complex number

✓ states the correct argument for $arg(w+2)$

Question 15 **(10 marks)**

A four metre long water tank, open at the top, is in the shape of a triangular prism. The triangular face is a right isosceles triangle with congruent sides of one metre length.



Initially the tank is completely full with water, but it develops a leak and loses water at a constant rate of 0.08 cubic metres per hour.

Let $h =$ the depth of water, in metres, in the tank after t hours.

(a) Show that the volume of water in the tank V cubic metres, is given by the expression $V(h) = 4h^2$. (2 marks)

Solution
$h = x\cos 45^o$ i.e. $x = \sqrt{2}h$ Volume $V = \frac{1}{2}(x^2)(4) = 2x^2 = 2(\sqrt{2}h)^2$ i.e. $V(h) = 4h^2$ **or** Area of triangle $A = \frac{1}{2}bh$ where $b = 2h$ Volume $V = \frac{1}{2}(2h)(h)(4) = 4h^2$
Specific behaviours
✓ uses an appropriate method to relate dimensions ✓ forms the area of the triangular base correctly

(b) Determine the rate of change of the depth, correct to the nearest 0.01 metres per hour, when the depth is 0.6 metres. (3 marks)

Solution
$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ i.e. $-0.08 = 8h \times \frac{dh}{dt}$ $-0.08 = 8(0.6)\times\frac{dh}{dt}$ $\therefore \frac{dh}{dt} = -0.02$ m/hr (two decimal places) Hence the depth is decreasing at approximately 2 cm per hour when the depth is 0.6 metres.
Specific behaviours
✓ uses the chain rule correctly to relate the volume and depth rates ✓ substitutes the values for $\frac{dV}{dt}$ and h correctly ✓ calculates the depth rate with the correct units (no penalty for incorrect rounding)

Assume that the rate of leakage stays constant at 0.08 cubic metres per hour.

(c) Show that the differential equation that relates $\frac{dh}{dt}$ with the depth h is given by

$\frac{dh}{dt} = -\frac{1}{100h}$. (1 mark)

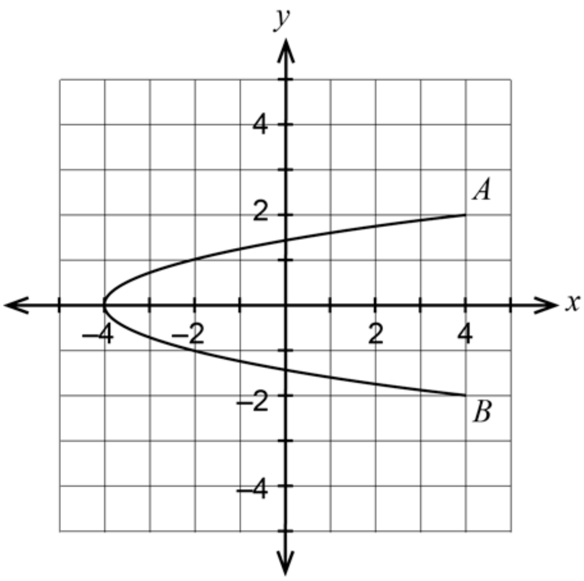
Solution
From $-0.08 = 8h \times \frac{dh}{dt}$ Differential equation : $\frac{dh}{dt} = -\frac{0.01}{h} = -\frac{1}{100h}$
Specific behaviours
✓ forms the differential equation correctly

(d) Hence determine the relationship for the depth h at any time t hours. (4 marks)

Solution
When $t=0$, $h=\dfrac{1}{\sqrt{2}}$ From $\dfrac{dh}{dt} = -\dfrac{0.01}{h}$, $\int h\ dh = \int -0.01\ dt$ i.e. $\dfrac{h^2}{2} = -0.01t + c$ $\therefore \dfrac{\left(\frac{1}{\sqrt{2}}\right)^2}{2} = -0.01(0) + c$ $\therefore c = 0.25$ $\dfrac{h^2}{2} = -0.01t + 0.25$ $h^2 = -0.02t + 0.5$ $\therefore h(t) = \sqrt{0.5 - 0.02t}$
Specific behaviours
✓ determine the initial value for h correctly ✓ separates the variables correctly ✓ anti-differentiates correctly ✓ determines the relationship between h and t correctly

Question 16 **(10 marks)**

A particle's position vector $\underset{\sim}{r}(t)$ is given by $\underset{\sim}{r}(t) = \begin{pmatrix} 4\cos 2t \\ 2\cos t \end{pmatrix}$ centimetres where t is measured in seconds. A plot of the path of the particle is shown below.



(a) Express the path of the particle as a Cartesian equation. (3 marks)

Solution
$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4\cos 2t \\ 2\cos t \end{pmatrix} \quad \therefore \quad x = 4\cos 2t, \quad y = 2\cos t$ i.e. $\cos t = \dfrac{y}{2}$ $\therefore \quad x = 4\left(2\cos^2 t - 1\right) = 8\cos^2 t - 4$ $\therefore \quad x = 8\left(\dfrac{y}{2}\right)^2 - 4$ i.e. $x = 2y^2 - 4$ where $-2 \le y \le 2$ and/or $-4 \le x \le 4$
Specific behaviours
✓ uses the cosine double angle identity correctly to eliminate t correctly ✓ obtains the correct Cartesian equation ✓ states a domain or range restriction correctly

(b) Determine the speed of the particle, correct to 0.01 cm per second, when it first reaches the point where $x = -2$. (4 marks)

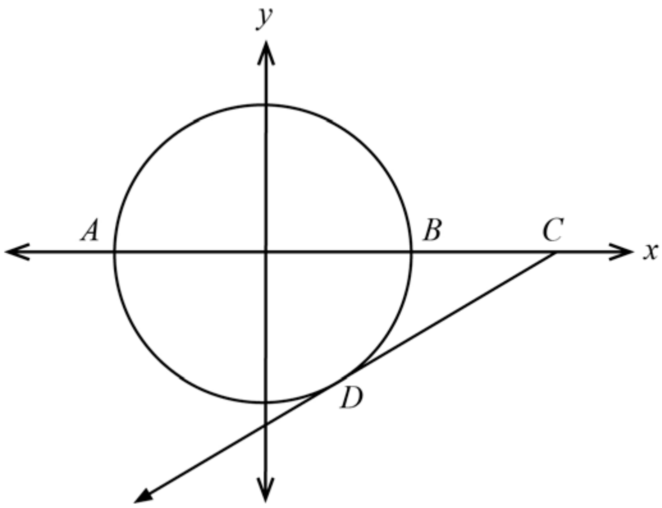
Solution
When $x = -2$ $\quad -2 = 4\cos 2t$ $\quad$ i.e. $\cos 2t = -\frac{1}{2}$ $\quad$ i.e. $2t = \frac{2\pi}{3}$ i.e. when $t = \frac{\pi}{3}$ $= 1.047...$ sec $\underset{\sim}{v}(t) = \frac{d\underset{\sim}{r}}{dt} = \begin{pmatrix} -8\sin 2t \\ -2\sin t \end{pmatrix} \quad \therefore \quad \underset{\sim}{v}\left(\frac{\pi}{3}\right) = \begin{pmatrix} -4\sqrt{3} \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} -6.928.. \\ -1.732.. \end{pmatrix}$ Speed $= \left\lvert \underset{\sim}{v}\left(\frac{\pi}{3}\right) \right\rvert = \sqrt{\left(4\sqrt{3}\right)^2 + \left(\sqrt{3}\right)^2}$ $= \sqrt{51} =$ 7.14 cm/sec (two decimal places)
Specific behaviours
✓ determines the value of t when $x = -2$ ✓ differentiates correctly to determine the velocity vector ✓ states that the speed is the magnitude of velocity ✓ evaluates the speed correctly (no penalty for incorrect rounding)

(c) Write the expression, in terms of trigonometric functions, for the distance the particle will travel along its path in travelling from point A to point B. Do **not** evaluate this expression. (3 marks)

Solution
At point A $(4,2)$ $\; t = 0$, at point B $(4,-2)$ $\; t = \pi$. Distance along the path $= \int_0^{\pi} \left\lvert \frac{d\underset{\sim}{r}}{dt} \right\rvert dt = \int_0^{\pi} \left\lvert \underset{\sim}{v}(t) \right\rvert dt$ $= \int_0^{\pi} Speed(t)\; dt$ $= \int_0^{\pi} \sqrt{64\sin^2 2t + 4\sin^2 t}\; dt$
Specific behaviours
✓ determine the correct values of t for points A, B ✓ writes a definite integral using the lower limit $t(A)$ and upper limit $t(B)$ ✓ writes the expression for the speed function correctly in terms of t

Question 17 **(8 marks)**

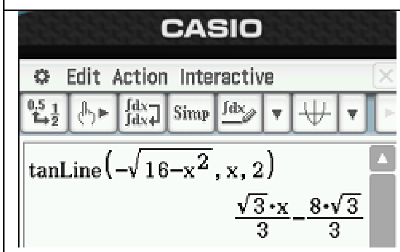
The diagram shows a circle with equation $x^2 + y^2 = 16$ with points A, B being the horizontal intercepts of this circle. DC is the tangent to the circle at point D, intersecting the x axis at point C. Point D has coordinates $\left(2, \ -2\sqrt{3}\right)$.



(a) Show that the equation for the tangent at point D can be written in the form $\sqrt{3}y = x - 8$. (3 marks)

Solution
$\frac{d}{dx}\left(x^2+y^2\right) = 0$ i.e. $2x + 2y\frac{dy}{dx} = 0$ i.e. $\frac{dy}{dx} = -\frac{x}{y}$
$\therefore \ m_{DC} = -\frac{2}{\left(-2\sqrt{3}\right)} = \frac{1}{\sqrt{3}}$ Equation tangent : $y + 2\sqrt{3} = \frac{1}{\sqrt{3}}\left(x-2\right)$
i.e. $\sqrt{3}y + 6 = \left(x-2\right)$ (multiplying each side by $\sqrt{3}$)
i.e. $\sqrt{3}y = x - 8$
Specific behaviours
✓ differentiates the circle equation correctly
✓ determines the gradient correctly
✓ forms the equation for the tangent correctly

or

Alternative solution

i.e. $\sqrt{3}y = x - 8$
Specific behaviours
✓✓ uses tanline on CAS correctly to find the tangent equation
✓ writes the equation for the tangent in the correct form

(b) Determine the coordinates of point C. (1 mark)

Solution
At point C, $y=0$ $\therefore$ $\sqrt{3}(0)=x-8$ $\therefore$ $x=8$ i.e. C has coordinates $(8,0)$.
Specific behaviours
✓ determines the correct x coordinate

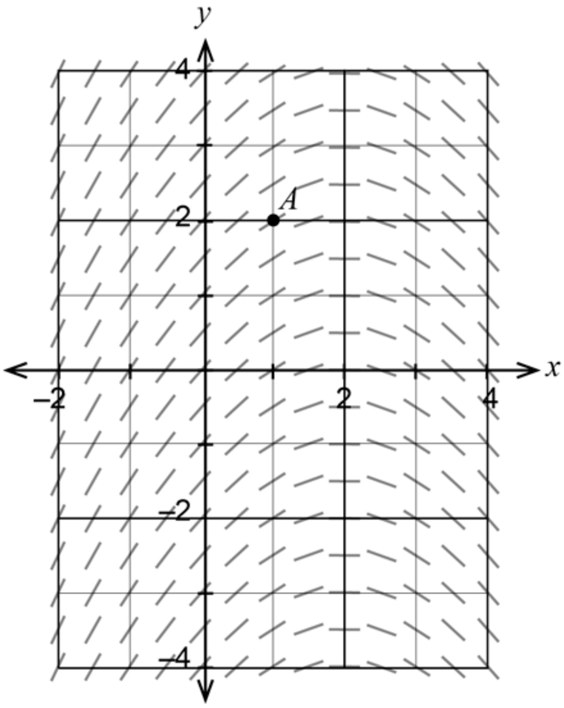
The region bounded by the arc AD, the line segment $\overline{DC}$ and the x axis is rotated about the x axis.

(c) Determine the volume of the resulting solid, correct to the nearest 0.01 cubic units. (4 marks)

Solution
Volume $= \int_{-4}^{2} \pi\left(16-x^2\right)dx + \int_{2}^{8} \pi\left(\frac{x-8}{\sqrt{3}}\right)^2 dx$ $= \int_{-4}^{2} \pi\left(16-x^2\right)dx + \int_{2}^{8} \frac{\pi}{3}(x-8)^2 dx$ $= 72\pi + 24\pi$ $= 96\pi$ $= 301.59$ cubic units (two decimal places)
Specific behaviours
✓ writes a sum of two integrals with the correct limits ✓ determines the correct integrand for the first integral ✓ determines the correct integrand for the second integral ✓ calculates the volume correct to 0.01 cubic units

Question 18 **(7 marks)**

A first-order differential equation has a slope field as shown in the diagram below.



(a) Determine the general differential equation that would yield this slope field. (3 marks)

Solution
General differential equation $\dfrac{dy}{dx} = a - bx$ where $a, b > 0$
Specific behaviours
✓ writes a linear function for the slope field ✓ indicates that the linear function has a positive vertical intercept ✓ indicates that the linear function has a negative gradient

The slope field at point $A\ (1,2)$ has a value of 0.5.

(b) Determine the equation for the curve $y = f(x)$ containing point A. (4 marks)

Solution
Clearly for $x = 2$, $\frac{dy}{dx} = 0 = a - b(2)$ i.e. $a = 2b$ Given $x = 1$, $\frac{dy}{dx} = 0.5 = a - b(1)$ i.e. $a - b = 0.5$ Solving gives $a = 1,\ b = 0.5$ $\therefore \frac{dy}{dx} = 1 - 0.5x$ Hence $y = x - \frac{x^2}{4} + c$ Since $(1,2) \in f$ then $2 = 1 - \frac{1}{4} + c$ i.e. $c = \frac{5}{4}$ $\therefore y = x - \frac{x^2}{4} + \frac{5}{4}$ is the equation containing point A.
Specific behaviours
✓ forms equations to represent the given slope values   ✓ determines the values for a, b i.e. determines the slope field equation   ✓ anti-differentiates correctly   ✓ determines the specific equation for $y = f(x)$ containing A

Question 19 **(16 marks)**

The volume of water used by the SavaDaWater company to top up an ornamental pool has been observed to be normally distributed with mean $\mu =$ 175 litres and standard deviation $\sigma =$ 15 litres.

The ornamental pool is topped up 50 times. Determine the probability that the:

(a) sample mean volume will be between 173 and 177 litres. (3 marks)

Solution
Let $\overline{W} =$ the sample mean from 50 times the pool is topped up (litres)
$= N\left(175,\ \sigma_{\overline{W}}^2\right)$ where $\sigma_{\overline{W}} = \dfrac{15}{\sqrt{50}} = 2.1213....$
Require $P\left(173 < \overline{W} < 177\right) = 0.6542$
Specific behaviours
✓ states that the sample mean is a normal random variable ✓ states the correct parameters for the normal random variable ✓ determines the correct probability

(b) total volume of water used is less than 8.96 kilolitres. (3 marks)

Solution
For a total of 8.96 kL, the sample mean $\overline{W} = \dfrac{8960}{50} = 179.2$ litres
Require $P\left(\overline{W} < 179.2\right) = 0.9761$
Specific behaviours
✓ calculates the sample mean correctly for the total 8.96 kL ✓ writes the correct event in terms of the required sample mean ✓ determines the correct probability

Water is a scarce commodity and accuracy is required. The pool is topped up 50 times and the sample mean obtained is denoted by $\overline{W}$.

(c) If it is required that $P\left(a \le \overline{W} \le b\right) = 0.99$, then determine the values of a and b, each correct to 0.1 litres. (3 marks)

Solution
As $\overline{W} = N\left(175,\ \sigma_{\overline{W}}^2\right)$ where $\sigma_{\overline{W}} = \dfrac{15}{\sqrt{50}} = 2.1213....$ Given $P\left(-k < z < k\right) = 0.99$, $k = 2.5758$ Interval : $175 - 2.5758\left(\sigma_{\overline{W}}\right) < \overline{W} < 175 + 2.5758\left(\sigma_{\overline{W}}\right)$ i.e. $169.536 < \overline{W} < 180.464$ i.e. the sample mean 99% confidence interval is 169.5 litres to 180.5 litres
Specific behaviours
✓ uses the correct parameters for the distribution of $\overline{W}$ ✓ determines the value for k for the confidence interval ✓ states the interval correct to 0.1 litres

(d) If the probability for the mean amount of water used differs from μ by less than five litres is 96%, find n, the number of waterings that need to be measured. (3 marks)

Solution
$\sigma_{\overline{W}} = \dfrac{15}{\sqrt{n}}$ Require $P\left(-k < z < k\right) = 0.96$ $\therefore\ k = 2.0537$ Hence $2.0537\left(\dfrac{15}{\sqrt{n}}\right) < 5$ Solving gives $n > 37.96$ i.e. we require at least 38 waterings to have the mean differ by less than five litres
Specific behaviours
✓ determines the standard z score that represents 96 % confidence ✓ forms the correct inequality to solve for n ✓ states the correct minimum integer value for n

A rival company called WolliWorks takes over the watering of the ornamental pool. Over 36 consecutive days, it was observed that the WolliWorks company used a total of 6.57 kilolitres. The standard deviation for the 36 days was also 15 litres.

A representative from the SavaDaWater company states that 'WolliWorks are using significantly more water than we did when we were watering this pool. They are wasting water'.

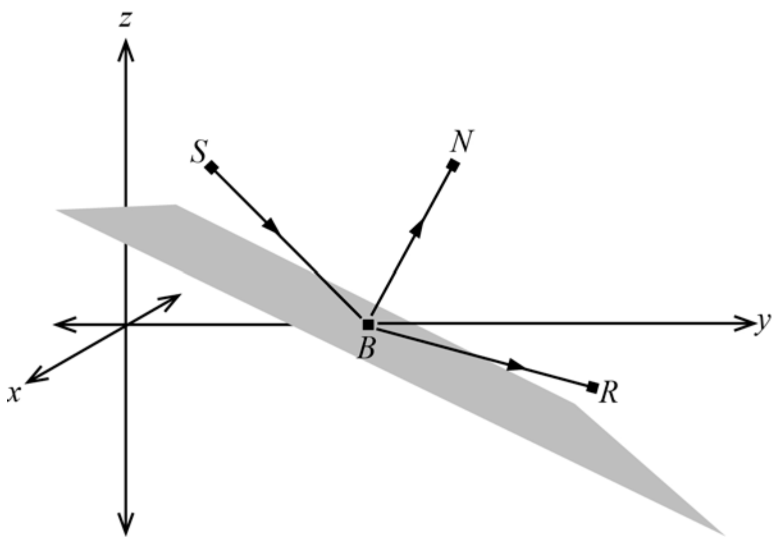
(e) Perform the calculations necessary to comment on this claim. (4 marks)

Solution
Let $\mu_w =$ the population mean for the Waterworks company (litres) For the WolliWorks total of 6 570 litres, this gives $\overline{W} = 182.5$ litres We will estimate μ_w as $N\left(182.5,\ \sigma_{\overline{W}}^2\right)$ where $\sigma_{\overline{W}} = \dfrac{15}{\sqrt{36}} = 2.5$ Confidence Interval for μ_w 95% level $182.5 - 1.96\left(\sigma_{\overline{W}}\right) < \mu_w < 182.5 + 1.96\left(\sigma_{\overline{W}}\right)$ i.e. $177.6 < \mu_w < 187.4$ Confidence Interval for μ_w 99% level $182.5 - 2.58\left(\sigma_{\overline{W}}\right) < \mu_w < 182.5 + 2.58\left(\sigma_{\overline{W}}\right)$ i.e. $176.0 < \mu_w < 189.0$ The SavaDaMoney population mean $\mu =$ 175 is outside the confidence interval using $\overline{W} = 182.5$ and $\sigma = 15$. i.e. the claim is vindicated. i.e. the WolliWorks company IS using significantly more water. Whether they are wasting water cannot be determined from the given data.
Specific behaviours
✓ determines the expected variation using $n = 36$ ✓ determines an appropriate confidence interval for the WolliWorks population mean ✓ states that the SavaDaMoney population mean 175 is outside the confidence interval ✓ concludes correctly by writing a comment about the claim

Question 20 **(7 marks)**

A laser pointer at point S directs a highly focused beam of light towards a mirror. The beam bounces off the mirror at point B and is then reflected away from the mirror toward point R.

The mirror's surface is given by the equation $\underset{\sim}{r} \cdot \left(\underset{\sim}{j} + 2\underset{\sim}{k}\right) = 9$ and the laser pointer is positioned at point S with position vector $-2\underset{\sim}{i} + 3\underset{\sim}{j} + 6\underset{\sim}{k}$. The laser pointer is held so that the beam is pointed in the direction $\underset{\sim}{d}_1 = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$.



(a) Determine the position vector for point B. (4 marks)

Solution
Equation for path $\overrightarrow{SB}$: $\underset{\sim}{r} = \begin{pmatrix} -2 \\ 3 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2+\lambda \\ 3+\lambda \\ 6-\lambda \end{pmatrix}$ Intersection of $\overrightarrow{SB}$ with mirror plane $y + 2z = 9$: $\begin{pmatrix} -2+\lambda \\ 3+\lambda \\ 6-\lambda \end{pmatrix} . \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 9$ i.e. $1(3+\lambda) + 2(6-\lambda) = 9$ Solving gives $\lambda = 6$.   Hence point B has position vector $\begin{pmatrix} 4 \\ 9 \\ 0 \end{pmatrix}$ i.e. $4\underset{\sim}{i} + 9\underset{\sim}{j}$.
Specific behaviours
✓ determines the vector equation for the incoming laser beam   ✓ substitutes correctly into the equation of the plane   ✓ solves correctly to determine the value for λ ✓ determines the position vector for point B

The laser beam is reflected away from the mirror so that:

- the angle of the incoming beam $\overrightarrow{SB}$ to the normal of the mirror is equal to the angle of the reflected beam $\overrightarrow{BR}$ to the normal of the mirror i.e. $s\angle SBN = s\angle NBR$.
- the incoming beam $\overrightarrow{SB}$, the normal of the mirror and the reflected beam $\overrightarrow{BR}$ are all contained in one plane.

Let $\hat{\underset{\sim}{d}}_2$ = the unit vector in the direction of the reflected beam $\overrightarrow{BR}$ i.e. $\left|\hat{\underset{\sim}{d}}_2\right| = 1$.

(b) Determine the unit vector $\hat{\underset{\sim}{d}}_2$ giving components correct to 0.01. (3 marks)

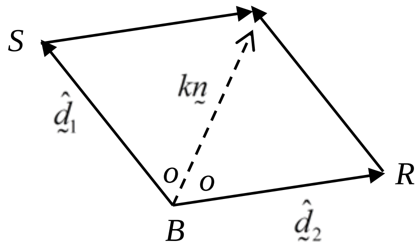
Solution

Let $\hat{\underset{\sim}{d}}_1$ = the unit vector in the direction of the incoming beam $\overrightarrow{SB}$

$\underset{\sim}{n}$ = the normal vector for the mirror surface

The normal vector $\underset{\sim}{n}$ bisects $\angle SBR$.

Consider the rhombus formed using vectors $\hat{\underset{\sim}{d}}_1$ and $\hat{\underset{\sim}{d}}_2$ as the adjacent sides:



Vector $\hat{\underset{\sim}{d}}_2$ is such that $\hat{\underset{\sim}{d}}_1 + \hat{\underset{\sim}{d}}_2 = k\underset{\sim}{n}$

i.e. $\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\-1\\1\end{pmatrix} + \hat{\underset{\sim}{d}}_2 = k\begin{pmatrix}0\\1\\2\end{pmatrix}$ i.e. $\hat{\underset{\sim}{d}}_2 = \begin{pmatrix}\frac{1}{\sqrt{3}}\\ k+\frac{1}{\sqrt{3}}\\ 2k-\frac{1}{\sqrt{3}}\end{pmatrix}$

As $\left|\hat{\underset{\sim}{d}}_2\right| = 1$ then $\frac{1}{3} + \left(k+\frac{1}{\sqrt{3}}\right)^2 + \left(2k-\frac{1}{\sqrt{3}}\right)^2 = 1^2$

Solving gives $k = \frac{2}{5\sqrt{3}}$ (rejecting $k = 0$)

$\therefore \hat{\underset{\sim}{d}}_2 = \begin{pmatrix}\frac{1}{\sqrt{3}}\\ \frac{7}{5\sqrt{3}}\\ -\frac{1}{5\sqrt{3}}\end{pmatrix} = \begin{pmatrix}0.58\\0.81\\-0.12\end{pmatrix}$ (two decimal places)

Specific behaviours

✓ indicates the formation of a rhombus using the unit vectors and the normal vector
✓ forms the equation to solve for the multiple k of the normal vector
✓ determines the components for $\hat{\underset{\sim}{d}}_2$

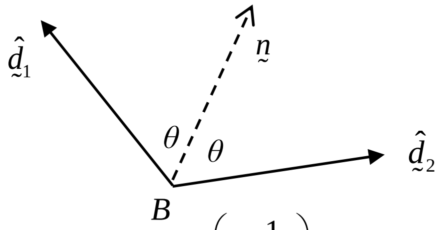
(b) Determine the unit vector $\hat{\underset{\sim}{d}}_2$ giving components correct to 0.01. (3 marks)

Alternative Solution

Let $\hat{\underset{\sim}{d}}_1 =$ the unit vector in the direction of the incoming beam $\overrightarrow{SB}$

The normal vector $\underset{\sim}{n}$ bisects $\angle SBR$. Let $\theta = s\angle SBN = s\angle NBR$

Examine the dot product between vectors:



$\hat{\underset{\sim}{d}}_1 \cdot \underset{\sim}{n} = \left|\hat{\underset{\sim}{d}}_1\right|\left|\underset{\sim}{n}\right|\cos\theta \quad \therefore \quad \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} . \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 1\left(\sqrt{5}\right)\cos\theta$ i.e. $\cos\theta = \frac{1}{\sqrt{15}}$

Let $\hat{\underset{\sim}{d}}_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ such that $a^2 + b^2 + c^2 = 1 \quad \ldots (1)$.

$\hat{\underset{\sim}{d}}_2 \cdot \underset{\sim}{n} = \left|\hat{\underset{\sim}{d}}_2\right|\left|\underset{\sim}{n}\right|\cos\theta \quad \therefore \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} . \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 1\left(\sqrt{5}\right)\cos\theta$ i.e. $b + 2c = \sqrt{5}\cos\theta$

i.e. $b + 2c = \frac{1}{\sqrt{3}} \quad \ldots(2)$

As $\hat{\underset{\sim}{d}}_2 \in$ Plane then $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = x\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -x \\ y-x \\ x+2y \end{pmatrix}$ where $x, y \in \mathbb{R}$

Writing $(1),(2)$ in terms of x, y: $(-x)^2 + (y-x)^2 + (x+2y)^2 = 1 \quad \ldots (3)$

$(y-x) + 2(x+2y) = \frac{1}{\sqrt{3}} \quad \ldots (2)$

Using CAS : Solving gives $x = -0.5773,\ y = 0.2309$

(Reject $x = 0.5773,\ y = 0$ since this yields $\hat{\underset{\sim}{d}}_2 = \hat{\underset{\sim}{d}}_1$)

Hence $\hat{\underset{\sim}{d}}_2 = \begin{pmatrix} 0.58 \\ 0.81 \\ -0.12 \end{pmatrix}$ (two decimal places)

Specific behaviours

✓ forms the equation (2) using the dot product condition
✓ forms the equation (3) to solve for the multiple x, y
✓ determines the components for $\hat{\underset{\sim}{d}}_2$

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